Homework 2 solutions

Section 2.1: Ex 1,2,3,4,6,11; AP 1
(18 points; Ex 3, AP 1 graded, 4 pts each; 2 pts to try the others)

1. Determine if each function has a unique fixed point on the specified interval.
   (a) \( g(x) = 1 - x^2/4 \) on \([0,1]\). \( g'(x) = -x/2 \), so \( g \) is continuous and decreasing on \([0,1]\).
   \( g(0) = 1 \) and \( g(1) = \frac{1}{4} \). Therefore \( g(x) \) is in \([0,1]\) for all \( x \) in \([0,1]\). Furthermore,
   \[ |g'(x)| = \frac{|x|}{2} \leq 1/2 < 1 \] for \( x \) in \([0,1]\). The fixed point theorem thus implies the existence of an unique fixed point in \([0,1]\).

   (b) \( g(x) = 2^x \) on \([0,1]\). \( g'(x) = -\log(2) \cdot 2^x \), so \( g \) is continuous and decreasing on \([0,1]\).
   \( g(0) = 1 \) and \( g(1) = 1/2 \). Therefore \( g(x) \) is in \([0,1]\) for all \( x \) in \([0,1]\). And since \( g \) is decreasing, there is a unique fixed point in \([0,1]\).

   (c) \( g(x) = 1/x \) on \([0.5, 5.2]\). \( g(x) = x \) is \( 1/x = x \), or \( x^2 = 1 \), which has a single solution \( x = 1 \) in the interval \([0.5, 5.2]\).

2. \( g(x) = -4 + 4x - \frac{1}{2} x^2 \).
   (a) Show that \( P=2 \) and \( P=4 \) are fixed points. We solve \( g(x) = x \):
   \[-4 + 4x - \frac{1}{2} x^2 = x, \]
   \[ \frac{1}{2} x^2 - 3x + 4 = 0, \]
   \[ x^2 - 6x + 8 = 0, \]
   \[ (x - 4)(x - 2) = 0, \]
   which has solutions \( x=2 \) and \( x=4 \).

   (b) Use the starting value \( p_0 = 1.9 \) and compute \( p_1, p_2 \), and \( p_3 \).
   Using the iteration \( p_{n+1} = g(p_n) \),
   \[ p_1 = g(p_0) = -4 + 4(1.9) - \frac{1}{2} (1.9)^2 = 1.795 \]
   \[ p_2 = g(p_1) = -4 + 4(1.795) - \frac{1}{2} (1.795)^2 = 1.569 \]
   \[ p_3 = g(p_2) = -4 + 4(1.569) - \frac{1}{2} (1.569)^2 = 1.045 \]

   (c) Use the starting value \( p_0 = 3.8 \) and compute \( p_1, p_2 \), and \( p_3 \).
   Using the iteration \( p_{n+1} = g(p_n) \),
   \[ p_1 = g(p_0) = -4 + 4(3.8) - \frac{1}{2} (3.8)^2 = 3.98 \]
   \[ p_2 = g(p_1) = -4 + 4(3.98) - \frac{1}{2} (3.98)^2 = 3.9998 \]
   \[ p_3 = g(p_2) = -4 + 4(3.9998) - \frac{1}{2} (3.9998)^2 = 4.0000 \]

   (d) Find the errors \( E_k \) and relative errors \( R_k \) for the values \( p_k \) in parts (b) and (c).
   For (b): \[ E_1 = |1.795 - 2| = .205, \quad R_1 = |1.795-2|/2 = .1025 \]
   \[ E_2 = |1.569 - 2| = .431, \quad R_1 = |1.569-2|/2 = .2155 \]
   \[ E_3 = |1.045 - 2| = .955, \quad R_1 = |1.045-2|/2 = .4775 \]

   For (c): \[ E_1 = |3.98 - 4| = .02, \quad R_1 = |3.98-4|/4 = .005 \]
   \[ E_2 = |3.9998-4| = .0002, \quad R_1 = |3.9998-4|/4 = .00005 \]
   \[ E_3 = |4.0000-4| = 2 \times 10^{-8}, \quad R_1 = |4.0000-4|/4 = 5 \times 10^{-9} \]
What conclusions can be drawn from Theorem 2.3?

\[ g'(x) = 4 - x. \]

\[ g'(2) = 2, \quad g'(4) = 0 \]

The derivative at 2 is greater than 1, so \( x^* = 2 \) is an unstable fixed point.

The derivative at 4 is less than 1, so \( x^* = 4 \) is a locally stable fixed point.

3. Graph the function with the line \( y=x \), construct a cobweb diagram, and determine if the fixed point iteration converges.

(a) \( g(x) = (6 + x)^{1/2}, \ P = 3, \) and \( p_0 = 7. \)

(b) \( g(x) = 1 + 2/x, \ P = 2, \) and \( p_0 = 4. \)

(c) \( g(x) = x^2/3, \ P = 3, \) and \( p_0 = 3.5. \)

(d) \( g(x) = -x^2 + 2x + 2, \ P = 2, \) and \( p_0 = 2.5. \)

The diagrams are shown below.

(a) The iteration converges monotonically.

(b) The iteration exhibits oscillating convergence.

(c) The iteration does not converge—it diverges to infinity.

(d) The iteration does not converge.
4. Let \( g(x) = x^2 + x - 4 \). Can fixed point iteration be used to find the solution to the equation \( g(x) = x \)? Why?

First, find the fixed point by solving \( g(x) = x \):
\[
\begin{align*}
x^2 + x - 4 &= x \\
x^2 - 4 &= 0 \\
x &= 2, \text{ or } x = -2
\end{align*}
\]
To determine if the iteration converges, we need to find the derivative at the fixed points.
\[
g'(x) = 2x + 1
\]
\[
g'(2) = 5, \quad g'(-2) = -3
\]
Since \(|g'(2)|\) and \(|g'(-2)|\) are both >1, the iteration will not converge.

6. Suppose \( g(x) \) and \( g'(x) \) are continuous on \((a, b)\); \( p_0, p_1, p_2 \) are in \((a, b)\), and \( p_1 = g(p_0) \) and \( p_2 = g(p_1) \). Also assume that there exists a constant \( K \) such that \(|g'(x)| < K\). Show that \(|p_2 - p_1| < K|p_1 - p_0|\). Hint: Use the mean value theorem.

We have \( p_2 - p_1 = g(p_1) - g(p_0) = g'(c_0)(p_1 - p_0) \) for some \( c_0 \) between \( p_0 \) and \( p_1 \) (by the mean value theorem)

Since \( p_0 \) and \( p_1 \) are in \((a, b)\), \( c_0 \) must also be in \((a, b)\), and therefore \(|g'(c_0)| < K\).

Therefore,
\[
|p_2 - p_1| = |g'(c_0)(p_1 - p_0)| = |g'(c_0)||p_1 - p_0| < K|p_1 - p_0|.
\]

11. For fixed-point iteration, discuss why it is an advantage to have \( g'(P) = 0 \).

The iteration satisfies \(|p_{n+1} - p_n| < K|p_n - p_{n-1}|\), where \( K \) is a bound on the absolute value of the derivative in the region of the fixed point. Thus, the smaller \(|g'(P)|\), the smaller will \( K \) in the inequality above be, and the closer will successive iterates be. This means that the iteration will converge faster if the derivative is smaller. The smallest derivative is 0, so an iteration around a fixed point with a zero derivative will converge rapidly.
ALGORITHMS AND PROGRAMS

1. Use the fixed point iteration algorithm to approximate the fixed points (if any) of the following functions to 12 decimal places. Produce a graph of each function and the line y=x that clearly shows any fixed points.

(a) \( g(x) = x^5 - 3x^3 - 2x^2 + 2 \)

The iteration does not converge. There are 3 unstable fixed points

(b) \( g(x) = \cos(\sin(x)) \)

\( x^* = 0.7681691567 \)

(c) \( g(x) = x^2 - \sin(x + 0.15) \)

The iteration does not converge.

(d) \( g(x) = x^{\cos(x)} \)

The iteration will to converge to the fixed point \( x^*=1 \) if the initial condition is in the interval \([.6, 1.2]\). There is another unstable fixed point at approximately 1.283.

Cobweb diagrams are below. MATLAB code follows.
```matlab
num=20;

% DEFINE FUNCTION USED IN ITERATIONS
f=@(x)cos(sin(x)); ic=-1; a=-1;b=2;c=-.1;d=1.1; fs=sprintf('cos(sin(x))');

err = abs(x(i+1)-x(i))/abs(x(i));
disp(sprintf('Fixed point at x = %.12f
Relative error: %.3e',x(i+1),err))
```