Homework 7 solutions

Section 3.7: Ex 3,4,10; AP 1,5,6
Section 4.2: Ex 3; AP 2

Section 3.7

3. Find a region in the xy-plane such that if \((p_0, q_0)\) is in the region then fixed-point iteration is guaranteed to converge for the system

\[
\begin{align*}
x &= g_1(x, y) = (x^2 - y^2 - x - 3)/3 \\
y &= g_2(x, y) = (x + y + 1)/3
\end{align*}
\]

We need to find conditions so that \(|\partial g_i/\partial x| + |\partial g_i/\partial y| < 1\) for \(i=1,2\) and \((x,y)\) a region containing the fixed point. We calculate the conditions as

\[
\left| \frac{\partial g_1}{\partial x} \right| + \left| \frac{\partial g_1}{\partial y} \right| = \left| (2x - 1)/3 \right| + \left| 2y/3 \right| < 1
\]

The 2\textsuperscript{nd} inequality is always true. The first defines a rectangular region in the xy-plane. Fixed point iteration will converge to the fixed point inside the rectangular region, which is \((-0.6515, 0.1743)\), if the initial condition is close enough to the fixed point. The following graph shows the curves \(g_1(x,y)=x\), \(g_2(x,y)=y\), and the rectangular region defined by the 1\textsuperscript{st} inequality. The fixed point is the intersection of the curves, shown as a star in the graph.

4. Rewrite the following system in fixed point form so that fixed-point iteration will converge to the solution.

\[
\begin{align*}
x + 4y + z &= 2 \\
x + y + 5z &= 0
\end{align*}
\]

We need to put the system into a form so that the sums of the rows of the matrix are less than 1. This can be done by performing the following operations on the rows:

Row 1: subtract 1 both sides, subtract 10x both sides, divide by -10
Row 2: subtract 2 both sides, subtract 5y both sides, divide by -5
Row 3: subtract 6z both sides, divide by -6
Then the system is in the form
\[-(-4x + y + z - 1)/10 = x\]
\[-(x - y + z - 2)/5 = y,\]
\[-(x + y - z)/6 = z\]
which satisfies the conditions for convergence of the fixed-point iteration.

10. Show that Newton’s method for two equations can be written in fixed-point iteration form
\[x = g_1(x, y), \quad y = g_2(x, y),\]
where
\[g_1(x, y) = x - \frac{f_1(x, y) \frac{\partial}{\partial y} f_2(x, y) - f_2(x, y) \frac{\partial}{\partial y} f_1(x, y)}{\det(J(x, y))}\]
\[g_2(x, y) = y - \frac{f_2(x, y) \frac{\partial}{\partial x} f_1(x, y) - f_1(x, y) \frac{\partial}{\partial x} f_2(x, y)}{\det(J(x, y))}\]

Newton’s method for solving
\[f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\]
is
\[\tilde{x}^{(n+1)} = \tilde{x}^{(n)} - [J(\tilde{x}^{(n)})]^{-1} f(\tilde{x}^{(n)})\], or, in two dimensions,
\[\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - [J(x_n, y_n)]^{-1} \begin{pmatrix} f_1(x_n, y_n) \\ f_2(x_n, y_n) \end{pmatrix}\]
As long as J is non-singular, we can calculate its inverse:
\[[J(x, y)]^{-1} = \begin{pmatrix} \frac{\partial}{\partial x} f_1(x, y) & \frac{\partial}{\partial y} f_1(x, y) \\ \frac{\partial}{\partial x} f_2(x, y) & \frac{\partial}{\partial y} f_2(x, y) \end{pmatrix}^{-1} = \frac{1}{\det(J(x, y))} \begin{pmatrix} \frac{\partial}{\partial y} f_2(x, y) & -\frac{\partial}{\partial y} f_1(x, y) \\ -\frac{\partial}{\partial x} f_2(x, y) & \frac{\partial}{\partial x} f_1(x, y) \end{pmatrix}\]
so the Newton iteration is
\[\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \frac{1}{\det(J(x_n, y_n))} \begin{pmatrix} \frac{\partial}{\partial y} f_2(x_n, y_n) & -\frac{\partial}{\partial y} f_1(x_n, y_n) \\ -\frac{\partial}{\partial x} f_2(x_n, y_n) & \frac{\partial}{\partial x} f_1(x_n, y_n) \end{pmatrix} \begin{pmatrix} f_1(x_n, y_n) \\ f_2(x_n, y_n) \end{pmatrix}\]
\[= \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \frac{1}{\det(J(x_n, y_n))} \begin{pmatrix} f_1(x_n, y_n) \frac{\partial}{\partial y} f_2(x_n, y_n) - f_2(x_n, y_n) \frac{\partial}{\partial y} f_1(x_n, y_n) \\ f_2(x_n, y_n) \frac{\partial}{\partial x} f_1(x_n, y_n) - f_1(x_n, y_n) \frac{\partial}{\partial x} f_2(x_n, y_n) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y} f_2(x_n, y_n) & -\frac{\partial}{\partial y} f_1(x_n, y_n) \\ -\frac{\partial}{\partial x} f_2(x_n, y_n) & \frac{\partial}{\partial x} f_1(x_n, y_n) \end{pmatrix}^{-1} \begin{pmatrix} f_1(x_n, y_n) \\ f_2(x_n, y_n) \end{pmatrix}\]
Algorithms and Programs

1. Use fixed point iteration and nonlinear Seidel iteration to approximate the fixed points of the following functions:

(a) \[ g_1(x, y) = \frac{8x - 4x^2 + y^2 + 1}{8}; \quad (x^*, y^*) = (1.1165151390, 1.9966031710) \]

We define the anonymous function as follows:

\[ g = @(x)[(8*x(1)-4*x(1)^2+x(2)^2+1)/8; ... \]


(b) \[ g_2(x, y) = \frac{2x - x^2 + 4y - y^2 + 3}{4}; \quad (x^*, y^*) = (-0.2695032304, -1.3154834723) \]

As above, we use

\[ g = @(x)[(x(2)-x(1)^3+3*x(1)^2+3*x(1))/7; (2*x(1)-x(1)^2+4*x(2)-x(2)^2+3)/4] \]

and we find the fixed point above.
5. For the following nonlinear system, sketch the graphs of both curves, and use the Newton iteration to find the points of intersection to nine decimal places.

\[ f_1(x, y) = 7x^3 - 10x - y - 1 = 0 \]
\[ f_2(x, y) = 8y^3 - 11y + x - 1 = 0 \]

There are nine solutions which we find as follows. We define the function \( f \) as,

\[ f=@(x)[7*x(1)^3-10*x(1)-x(2)-1; 8*x(2)^3-11*x(2)+x(1)-1] \]

We will also need the Jacobian, which is

\[ \text{Jac}=@(x)[21*x(1)^2-10 -1; 24*x(2)^2-11] \]

Then we can use the Newton iteration code as follows:

\[ [P, \text{iter}, \text{err}]=\text{newdim}(f, \text{Jac}, [1.5;0], 10^-10, 10^-10, 1000) \]

The above line finds the zero near \((1.5, 0)\), which is \((1.2434, 0.0221)\). To find the other solutions, we change our initial condition so that it is close to the solution we are attempting to find. The solutions and a graph of the curves are seen below.

Solutions:
\((-1.0604, 1.2569), (-0.2311, 1.2250), (1.2913, 1.1591), (-1.1531, -0.2017), (-0.0905, -0.0999), (1.2434, 0.0221), (-1.1980, -1.0558), (0.0125, -1.1248), (1.1861, -1.1810)\)

6. The above system can be rewritten in fixed point form as follows:

\[ x = (7x^3 - y - 1)/10 \]
\[ y = (8y^3 + x - 1)/11 \]

Discover that, regardless of the initial condition, only one of the nine solutions can be found by fixed-point iteration using this form. Are there other forms that could be used to find the other solutions?

Fixed point iteration converges to \((-0.0905, -0.0999)\) as long as the initial condition is close enough, but not to any of the other solutions. Yes, other forms of fixed point iteration can be used to find the other solutions. For example, the Newton iteration used in problem 5 is a fixed point iteration that can be used to find all nine of the solutions.
Section 4.2

3. Consider \( P(x) = -0.0292166667 x^3 + 0.275 x^2 - 0.570833333 x + 1.375 \), which passes through the four points (1, 1.05), (2, 1.10), (3, 1.35), and (5, 1.75).

(a) Show that the ordinates 1.05 1.10, 1.35, and 1.75 differ from those of Example 4.4 by less than 1.8%, yet the coefficients of \( x^3 \) and \( x \) differ by more than 42%.

The relative difference in the ordinates are:
- \( (1.05 - 1.05)/1.05 \approx -0.94\% \)
- \( (1.10 - 1.12)/1.12 \approx -1.79\% \)
- \( (1.35 - 1.34)/1.34 \approx 0.75\% \)
- \( (1.75 - 1.78)/1.78 \approx -1.69\% \)

The relative differences in the coefficients are:
- \( (-0.0292166667 + 0.02)/0.02 \approx 45.83\% \) change in \( a_3 \) and
- \( (-0.570833333 + 0.4)/0.4 \approx 42.71\% \) change in \( a_1 \).

(b) Find \( P(4) \) and compare with Example 4.4.
\( P(4) = 1.625 \), as compared with 1.60 in Example 4.4.

(c) Find \( P'(4) \) and compare with Example 4.4.
\( P'(4) = 0.229166667 \), as compared with 0.24 in Example 4.4.

(d) Find the definite integral of \( P(x) \) taken over \([1, 4]\) and compare with Example 4.4.
\[ \int_1^4 P(x) \, dx = 3.759375 \], as compared with 3.765 in Example 4.4.

(e) Find the extrapolated value \( P(5.5) \) and compare with Example 4.4.
\( P(5.5) = 1.70156261 \), as compared with 1.8025 in Example 4.4.

Algorithms and Programs

2. (i) For each of the given functions, find the 5th degree polynomial \( P(x) \) that passes through the 6 points \((x_j, f(x_j))\), where \( x_j = (j-1)/5 \) by solving the linear system \( P(x_j) = f(x_j) \) for \( j = 0, \ldots, 5 \). (ii) Compute the interpolated values \( P(0.3), P(0.4), P(0.5) \), and compare with \( f(0.3), f(0.4) \) and \( f(0.5) \). (iii) Compute the extrapolated values \( P(-0.1) \) and \( P(1.1) \), and compare with \( f(-0.1) \) and \( f(1.1) \). (iv) Compute and compare the integrals of \( P(x) \) and \( f(x) \) over \([0, 1]\). (v) Compare the values of \( P(x) \) and \( f(x) \) on the values \( x_k = k/100 \) for \( k = 1, \ldots, 100 \).

A code to find the coefficients by solving the linear system is seen below. The table shows the polynomial and comparisons with the function \( f(x) \). A plot of \( P(x) \) and \( f(x) \) follows for the three functions.

```
function a = Polylin(x,y)

% Finds coefficients of polynomial that interpolates N points (x,y)
% Input: x - abscissas of interpolated points
% y - column vector of ordinates of interpolated points
% Output: a - N dimensional row vector where
% a(j) is coefficient of x^N-j in N-1 degree polynomial
```
% \text{N-1} \quad \text{N-2}
% \text{P(x)} = a(1) x^{D-1} + a(2) x^{D-2} + \ldots + a(N-1) x + a(N)

\text{N} = \text{length(x)};
\text{M} = \text{zeros(N)};
\text{for } i=1:N
    \text{M}(i,:) = x(i).^{(N-1:-1:0)};
\text{end}
\text{a} = \text{uptrbk(M,y)}';

<table>
<thead>
<tr>
<th></th>
<th>f(x) = e^x</th>
<th>f(x) = \sin(x)</th>
<th>f(x) = (1+x)^{1.5x}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interpolating P(x)</td>
<td>0.0139 x^5 + 0.0349 x^4</td>
<td>0.0073 x^5 + 0.0016 x^4</td>
<td>0.3945 x^3 - 0.0717 x^3</td>
</tr>
<tr>
<td></td>
<td>+ 0.1704 x^3 + 0.4991 x^2</td>
<td>0.1676 x^3 + 0.0002 x^2</td>
<td>+ 0.7304 x^3 + 0.9415 x^2</td>
</tr>
<tr>
<td></td>
<td>1.001 x + 1.0000</td>
<td>1.0000 x</td>
<td>1.0052 x + 1.0000</td>
</tr>
<tr>
<td>Interpolated values:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P(0.3)</td>
<td>1.349858</td>
<td>0.295520</td>
<td>1.406410</td>
</tr>
<tr>
<td>f(0.3)</td>
<td>1.349859</td>
<td>0.295520</td>
<td>1.406457</td>
</tr>
<tr>
<td>P(0.4)</td>
<td>1.491825</td>
<td>0.389418</td>
<td>1.601693</td>
</tr>
<tr>
<td>f(0.4)</td>
<td>1.491825</td>
<td>0.389418</td>
<td>1.601693</td>
</tr>
<tr>
<td>P(0.5)</td>
<td>1.648722</td>
<td>0.479425</td>
<td>1.837153</td>
</tr>
<tr>
<td>f(0.5)</td>
<td>1.648721</td>
<td>0.479426</td>
<td>1.837117</td>
</tr>
</tbody>
</table>

| Extrapolated values:    |                          |                        |                     |
| P(-0.1)                 | 0.904815                 | -0.099828              | 0.908150            |
| f(-0.1)                 | 0.904837                 | -0.099833              | 0.909533            |
| P(1.1)                  | 3.004140                 | 0.891215               | 4.747593            |
| f(1.1)                  | 3.004166                 | 0.891207               | 4.749638            |

| Integrals:              |                          |                        |                     |
| \int_0^1 P(x) dx =      | 1.718282                 | 0.459698               | 2.050480            |
| \int_0^1 f(x) dx =      | 1.718282                 | 0.459698               | 2.050446            |
| Max|P(x_i) - f(x_i)|      | 0.000026                 | 0.000008            | 0.002045            |