Theorem 12.13 (Differentiation of $f(t)$) Let $f(t)$ and $f'(t)$ be continuous for $t \geq 0$ and be of exponential order. Then,

$$L(f'(t)) = sF(s) - f(0),$$

where

$$F(s) = L(f(t)).$$

Proof. Let $K$ be large enough that both $f(t)$ and $f'(t)$ are of exponential order $K$. If $\text{Re}(s) > K$, then $L(f'(t))$ is given by

$$L(f'(t)) = \int_0^\infty f'(t) e^{-st} dt.$$
Next, using integration by parts, we rewrite this equation as

\[
\mathcal{L}(f'(t)) = \lim_{R \to +\infty} \left[ f(t) e^{-st} \right]_{t=0}^{t=R} + s \int_{0}^{\infty} f(t) e^{-st} dt.
\]

As \( f(t) \) is of exponential order \( K \) and \( \text{Re}(s) > K \), we have

\[
\lim_{R \to +\infty} f(R) e^{-sR} = 0.
\]

Hence the preceding equation becomes

\[
\mathcal{L}(f'(t)) = -f(0) + s \int_{0}^{\infty} f(t) e^{-st} dt = sF(s) - f(0),
\]

proving the theorem.

\textbf{Corollary 12.1} If \( f(t), f'(t), \) and \( f''(t) \) are of exponential order, then

\[
\mathcal{L}(f''(t)) = s^2 f(s) - sf(0) - f'(0).
\]