Theorem 11.5 (Invariance of flow) Let

\[ F_1(w) = \Phi(u, v) + i\Psi(u, v) \]

denote the complex potential for a fluid flow in a domain \( G \) in the \( w \) plane, where the velocity is

\[ V_1(u, v) = F_1'(w). \]

If the function \( w = S(z) = u(x, y) + iv(x, y) \) is a one-to-one conformal mapping from a domain \( D \) in the \( z \) plane onto \( G \), then the composite function

\[ F_2(z) = F_1(S(z)) = \Phi(u(x, y), v(x, y)) + i\Psi(u(x, y), v(x, y)) \]

is the complex potential for a fluid flow in \( D \), where the velocity is

\[ V_2(x, y) = F_2'(z). \]

The situation is shown in Figure 11.48.

Proof From Equation (11-34), \( F_1(w) \) is an analytic function. Since the composition of analytic functions is analytic, \( F_2(z) \) is the required complex potential for an ideal fluid flow in \( D \).

We note that the functions

\[ \phi(x, y) = \Phi(u(x, y), v(x, y)) \quad \text{and} \quad \psi(x, y) = \Psi(u(x, y), v(x, y)) \]
are the new velocity potential and stream function, respectively, for the flow in $D$. A streamline or natural boundary curve

$$\psi(x, y) = K$$

in the $z$ plane is mapped onto a streamline or natural boundary curve

$$\Psi(u, v) = K$$

in the $w$ plane by the transformation $w = S(z)$. One method for finding a flow inside a domain $D$ in the $z$ plane is to conformally map $D$ onto a domain $G$ in the $w$ plane in which the flow is known.

For an ideal fluid with uniform density $\rho$, the fluid pressure $P(x, y)$ and speed $|V(x, y)|$ are related by the following special case of Bernoulli’s equation:

$$\frac{P(x, y)}{\rho} + \frac{1}{2} |V(x, y)| = \text{constant}.$$ 

Note that the pressure is greatest when the speed is least.