1. Write out the Picard iteration scheme. If possible, find the solution.

(a) \( x' = x + 2, \ x(0) = 2 \). \( u_0 = 2, \ u_{k+1}(t) = 2 + \int_0^t (u_k(s) + 2)ds \).

\[
\begin{align*}
  u_0 &= 2 \\
  u_1(t) &= 2 + \int_0^t (2 + 2)ds = 2 + 4t \\
  u_2(t) &= 2 + \int_0^t (2 + 4s + 2)ds = 2 + 4t + 2t^2 \\
  u_3(t) &= 2 + \int_0^t (2 + 4s + 2s^2 + 2)ds = 2 + 4t + 2t^2 + \frac{2}{3}t^3 \\
  &\quad \cdots \\
  u_k(t) &= 2 + 4 \sum_{j=1}^k \frac{t^j}{j!} = 4 \sum_{j=0}^k \frac{t^j}{j!} - 2
\end{align*}
\]

Thus, \( u_k(t) \to 4e^t - 2 \), which is the solution of the IVP.

(b) \( x' = x^{4/3}, \ x(0) = 0 \). \( u_0 = 0, \ u_{k+1}(t) = 0 + \int_0^t (u_k(s))^{4/3}ds \).

Clearly every iterate is zero \( u_k(t) = 0 \), and the unique solution of the IVP is \( x(t) = 0 \).

(c) \( x' = x^{4/3}, \ x(0) = 1 \). \( u_0 = 1, \ u_{k+1}(t) = 1 + \int_0^t (u_k(s))^{4/3}ds \).

\[
\begin{align*}
  u_0 &= 1 \\
  u_1(t) &= 1 + \int_0^t 1^{4/3}ds = 1 + t \\
  u_2(t) &= 1 + \int_0^t (1 + s)^{4/3}ds = 1 + \frac{3}{4}(1 + t)^{7/3} - 3/7 = \frac{4}{7} + \frac{3}{7}(1 + t)^{7/3} \\
  u_3(t) &= 1 + \int_0^t \left( \frac{4}{7} + \frac{3}{7}(1 + s)^{7/3} \right)^{4/3}ds
\end{align*}
\]

The solution to the IVP is \( x(t) = -27(t - 3)^{-3} \).

(d) \( x' = \cos x, \ x(0) = 0 \). \( u_0 = 0, \ u_{k+1}(t) = \int_0^t \cos (u_k(s))ds \).

\[
\begin{align*}
  u_0 &= 0 \\
  u_1(t) &= \int_0^t \cos(0)ds = t \\
  u_2(t) &= \int_0^t \cos(s)ds = \sin(t) \\
  u_3(t) &= \int_0^t \cos(\sin(s))ds
\end{align*}
\]

The solution to the IVP is \( x(t) = 2 \tan^{-1}(\tanh(t/2)) \).
(c) \( x' = \frac{1}{2x}, \ x(1) = 1, \ u_0 = 1, \ u_{k+1}(t) = 1 + \int_1^t \frac{1}{2u_k(s)} \, ds \)

\[
\begin{align*}
    u_0 & = 1 \\
    u_1(t) & = 1 + \int_1^t \frac{1}{2} \, ds = 1 + \frac{1}{2} (t - 1) = \frac{1}{2} (t + 1) \\
    u_2(t) & = 1 + \int_1^t \frac{1}{s+1} \, ds = 1 - \log(2) + \log(t + 1)
\end{align*}
\]

The solution to the IVP is \( x(t) = \sqrt{t} \).

2. Let \( A \) be an \( n \times n \) matrix. Show that the Picard iteration scheme gives the solution \( e^{tA}X_0 \).

Solution. Here \( F(X) = AX \), so the Picard iteration scheme is \( U_0 = X_0, \ U_{k+1} = X_0 + \int_0^t AU_k(s) \, ds \). The iterates are

\[
\begin{align*}
    U_0 & = X_0 \\
    U_1(t) & = X_0 + \int_0^t AX_0 \, ds = (I + tA)X_0 \\
    U_2(t) & = X_0 + \int_0^t A(1 + sA)X_0 \, ds = X_0 + tAX_0 + \frac{t^2}{2} A^2 X_0 = \left( I + tA + \frac{t^2}{2} A^2 \right) X_0 \\
    \vdots \\
    U_k(t) & = \sum_{j=0}^{k} \frac{(tA)^j}{j!} X_0,
\end{align*}
\]

which converges to \( \sum_{j=0}^{\infty} \frac{(tA)^j}{j!} X_0 = e^{tA}X_0 \).

4. Verify the linearity principle for linear, nonautonomous systems.

Solution. Consider the equation \( X' = A(t)X \). Suppose \( X \) and \( Y \) are two solutions and \( \alpha, \beta \) are constants. Then

\[
\frac{d}{dt} (\alpha X + \beta Y) = \alpha X' + \beta Y' = \alpha A(t)X + \beta A(t)Y = A(t)(\alpha X + \beta Y).
\]

Thus \( \alpha X + \beta Y \) is also a solution.

6. Discuss the existence and uniqueness of solutions to \( x' = x^a, \ x(0) = 0, \) where \( a > 0 \).

Solution. For any \( a > 0, \ x(t) = 0 \) is a solution. So the question is whether or not this is the only solution. (Existence is guaranteed.) Notice that when \( a \geq 1, \ f(x) = x^a \) is continuously differentiable, so the existence & uniqueness theorem tells us that \( x(0) = 0 \) is the unique solution in this case. However, when \( 0 < a < 1, \ f(x) \) is not continuously differentiable at \( x = 0 \), and separation of variables gives us an additional solution, \( x(t) = ((1-a)t)^{1/(1-a)} \). In fact, this can also be shifted to \( \tau \) for any \( \tau > 0 \), so there are infinitely many solutions when \( 0 < a < 1 \).
7. For the system \( X' = A(t)X \), let \( P(t) \) be a matrix of solutions satisfying \( P' = A(t)P \), \( P(0) = P_0 \). (Notice that \( P_0 \) is a matrix.) First, suppose \( A(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( 2 \times 2 \), and let \( X \) and \( Y \) be the columns of \( P \), so \( X' = AX \), \( Y' = AY \). We let \( W(t) = \det P(t) \). Then

\[
\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}, \quad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix},
\]

so

\[
W' = \frac{d}{dt} \det P(t) = d \frac{dt}{dt} \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = d (x_1 y_2 - x_2 y_1)
=x_1' y_2 + x_1 y_2' - x_2' y_1 - x_2 y_1' = (ax_1 + bx_2) y_2 + x_1 (cy_1 + dy_2) - (cx_1 + dx_2) y_1 - x_2 (ay_1 + by_2) = (a + d)(x_1 y_2 - x_2 y_2) = \text{tr}A(t)W
\]

So \( W' = (\text{tr}A)W \), which has the unique solution

\[
W(t) = W(0) \exp \left( \int_0^t \text{tr}A(s) ds \right) \quad \text{or} \quad \det P(t) = (\det P_0) \exp \left( \int_0^t \text{tr}A(s) ds \right).
\]

For the general case, suppose \( P' = A(t)P \). Then, since \( P'(t) = \lim_{h \to 0} \frac{1}{h} (P(t + h) - P(t)) \),

\[
P(t + h) = hA(t)P(t) + P(t) + o(h) = (I + hA(t))P(t) + o(h)
\Rightarrow \det P(t + h) = \det(I + hA(t)) \det P(t) + o(h) = (1 + h \cdot \text{tr}A + O(h^2)) \det P(t) + o(h)
\Rightarrow \frac{1}{h} (\det P(t + h) - \det P(t)) = (\text{tr}A) \det P(t) + O(h)
\]

where we have used the fact that \( \det(I + h \cdot A) = 1 + h \cdot \text{tr}A + O(h^2) \). Now take the limit as \( h \to 0 \) on both sides to get

\[
\frac{d}{dt} \det P(t) = (\text{tr}A(t))P(t)
\]

Solving this equation gives the desired result.

8. Here is an example of a differential equation for which there is no solution for any initial condition:

\[
x' = \begin{cases} 
1 & \text{if } x \text{ is rational} \\
-1 & \text{if } x \text{ is irrational}
\end{cases}
\]

For this equation \( x \) should be increasing at each rational number, but the slightest increase means it should pass an irrational number, at which it should be decreasing. This is impossible.