Throughout this assignment, $\mathbb{H}^2$ will refer to the upper half-plane; that is, the set $\{ (x, y) \in \mathbb{R}^2 \mid y > 0 \}$. Points in the model will refer to elements of this set. Lines will be systems of inequalities of the form $\{ x = c, y > 0 \}$ or $\{ (x - c)^2 + y^2 = r^2, y > 0 \}$. A point represented by an ordered pair is incident to a line represented by a system of inequalities if the ordered pair satisfies the inequality. If a path $\gamma(t) = (x(t), y(t))$ connects $\gamma(a) = (x_1, y_1)$ to $\gamma(b) = (x_2, y_2)$, then the length of that path can be computed as

$$\text{length}(\gamma) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$ 

The distance between two points is then interpreted as the length of the shortest (continuous) path between them.

1. Read pages 371-376.

2. Consider the function $R : \mathbb{H}^2 \to \mathbb{H}^2$ defined by $D(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)$.
   (a) Check that $R$ fixes the Euclidean unit circle centered at the origin.
   (b) Let $a \neq 0$. Describe what $R$ does to the Euclidean ray $\{ (x, ax) \mid ax > 0 \}$.
   (c) Show that $R$ sends points ‘inside’ the Euclidean unit circle centered at the origin to points ‘outside,’ and vice versa.
   (d) Suppose a path in $\mathbb{H}^2$ is parameterized by $\gamma(t) = (r(t) \cos(\theta(t)), r(t) \sin(\theta(t)))$ from $0 \leq t \leq 1$. With some calculation, one can show that its length is

$$\text{length}(\gamma(t)) = \int_0^1 \sqrt{\left(\frac{\theta'}{r(t)} (r(t) \cos(\theta(t)))\right)^2 + \left(\frac{\theta'}{r(t)} (r(t) \sin(\theta(t)))\right)^2} \, dt = \int_0^1 \sqrt{\frac{r'(t)^2 + r(t)^2 \theta'(t)^2}{r(t) \sin(\theta(t))}} \, dt.$$ 

Show that $R(\gamma(t))$ has the same length.

3. Recall that $T_c : \mathbb{H}^2 \to \mathbb{H}^2$ defined by $T_c(x, y) = (x + c, y)$ is an isometry. Recall also that $D_k : \mathbb{H}^2 \to \mathbb{H}^2$ defined by $D_k(x, y) = (kx, ky)$ is an isometry. Given any two points $(x_1, y_1)$ and $(x_2, y_2)$ in $\mathbb{H}^2$, show that $T_{x_2 - x_1} \circ D_{y_2} (x_1, y_1) = (x_2, y_2)$. In other words, given any two points in $\mathbb{H}^2$, there is an isometry taking one point to the other.

4. In $\mathbb{R}^2$, let $l$ and $m$ be two lines passing through the origin, and forming acute angle $\theta$. Show that the composition of reflections across these lines is a rotation of angle $2\theta$. (Hint: Without loss of generality, let $l$ be the $x$-axis, and let $m$ pass through the first and third quadrants. Consider, first, what happens to points $(\cos \phi, \sin \phi)$ on the unit circle.)