5.5 a) Prove field $\mathbb{K}$ is isomorphic to ring of all $2 \times 2$ matrices with $(0, 0)$ w/ $a \in \mathbb{K}$

**Hint:** $f(a) = (0, a)$

**Injective:** $a \neq b \implies f(a) \neq f(b)$

\[
\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}
\]

**Surjective:** $\exists \mathbf{x} \in \mathbb{K}, f(x) = \mathbf{y}$

\[
\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mathbf{y} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}
\]

$f(a + b) = f(a) + f(b)$

\[
f(a) + f(b) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a+b \end{pmatrix} = f(a+b)
\]

$f(ab) = f(a)f(b)$

\[
\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & ab \end{pmatrix}
\]

8) The function $f$ can be given by $f(x) = (a \ x) \ a \in \mathbb{R}$

**Homomorphism:** $f(a+b) = (a+b \ b) = (a \ b) + (b \ b)$

\[
\begin{pmatrix} a-b & a \\ 0 & ab \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}
\]

**Injective:** Suppose $f(a) = f(b)$

\[
f(a) = (a \ b) = (a \ b)
\]

$\implies a = b$
Given: \( f: \mathbb{Z} \to \mathbb{Z} \) is an isomorphism

**Claim:** \( f \) is the identity map.

Let \( f(1) = a \), then \( f(1+1) = f(1) + f(1) = 2a \)
\[
\begin{align*}
f(2) &= 2a \\
\end{align*}
\]

And \( f(2) = f(2 \cdot 1) = f(a \cdot a) = 2a^2 \)
\[
\begin{align*}
2a^2 &= 2a \\
\end{align*}
\]

\( a(a - 1) = 0 \)
\( a = 0 \) or \( a = 1 \)

But by Theorem 3.12, \( f(0) = 0 \) and \( 1 \neq 0 \), so since \( f \) is injective, \( f(1) \neq 0 \).
\( \therefore f(1) = 1 \).

Now for all \( n \in \mathbb{Z} \),
\[
\begin{align*}
f(n) &= f(1 + 1 + \ldots + 1) = f(1) + f(1) + \ldots + f(1) = n \quad (n \text{ times}) \\
\end{align*}
\]
And by Thm. 3.12, \( f(-n) = -f(n) = -n \).

Since \( f(1) = a \) for all \( a \in \mathbb{Z} \), \( f \) is the identity map.
(a) \[ f: \mathbb{Z} \rightarrow \mathbb{Z} \quad f(a) = -x \quad \text{not a hom.} \]

\[ \text{Let } a, b \in \mathbb{Z} \quad f(a + b) = -(a + b) = -x + y = f(a) + f(b) \]

\[ f(ab) = -ab = -ax - by = f(a)f(b) \]

If \( x \in \mathbb{Z}_2 \), then either \( x = [0] \) or \( x = [1] \).

- \([0] = [0] \) and \(-[1] = [1] \) in \( \mathbb{Z}_2 \).

(b) \( g: \mathbb{Q} \rightarrow \mathbb{Q} \quad g(x) = \frac{1}{x} \quad \text{not a hom.} \)

\[ \text{Let } a, b \in \mathbb{Q} \quad f(a + b) = \frac{1}{a + b} = \frac{1}{a} + \frac{1}{b} \]

\[ f(ab) = \frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b} = f(a)f(b) \]

(c) \( h: \mathbb{R} \rightarrow M(2, \mathbb{R}) \quad h(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \quad \text{not a hom.} \)

\[ \text{Let } a, b \in \mathbb{R} \quad h(a + b) = \begin{bmatrix} a + b & 0 \\ 0 & a + b \end{bmatrix} \neq \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = h(a) + h(b) \]

\[ h(ab) = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = h(a)h(b) \]

(d) \( i: \mathbb{Z}_{10} \rightarrow \mathbb{Z} \quad i(x) = [x] \quad i \text{ is a hom.} \)

\[ \text{Let } a, b \in \mathbb{Z}_{10} \quad [a + b] = [a] + [b] = i(a) + i(b) \]

\[ [ab] = [a] [b] = i(a)i(b) \]
(a) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( \mathbb{R} \) is the set of

2x2 matrices c.d. \( f(a) = (a, a) \) for \( a \in \mathbb{R} \)

\[ f(a) : \begin{pmatrix} a & b \\ 0 & a-b \end{pmatrix} = \begin{pmatrix} a & a \\ 0 & a-b \end{pmatrix} - \begin{pmatrix} b & b \\ 0 & b \end{pmatrix} - f(a) + f(b) \]

(b) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( f(a) = \sqrt{a} \) for \( a \in \mathbb{R} \)

\[ f(a \cdot b) = \sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \]

\[ f(ab) = (a \cdot b) \neq (a \cdot a) (b \cdot b) = (f(a) f(b)) \]

\[ f(ab) : \begin{pmatrix} a & b \\ 0 & a-b \end{pmatrix} = \begin{pmatrix} a & a \\ 0 & a-b \end{pmatrix} - \begin{pmatrix} b & b \\ 0 & b \end{pmatrix} - f(ab) + f(b) \]
13. If $a$ is not a zero divisor in $S$.

pf: $f : R \to S$ and $x$ is a zero divisor in $R$

therefore $x \not= 0_R$ and exist $b \in R$, $b \not= 0_R$

$rb = 0_R$

As $f$ is homomorphic,

$f(rb) = f(0_R) = 0_S$

$f(xb) = f(0_R) = 0_S$

but we do not know the value of $f(x)$ and $f(b)$

because we can assume that $f(x) = 0_s \in R$

therefore we can not define whether $f(x)$ is a divisor or not.

21. Let $R \times R$ is function $f$ that make $f(a \cdot b) = a + bi \cdot a \in R$

Let $c, d \in R$

$f(c \cdot d) = f((c \cdot d)) = f((c \cdot d)) = f((c + d) = f(c + d)$

$f((a, b), (c, d)) = f((a + c) \cdot (b + d)) = (a + b) \cdot (c + d)$

$f((a, b)) \cdot f((c, d)) = f((a \cdot c, b + d)) = f((a + b) \cdot (c + d)$

for $f((a, b)) = f((c, d))$

we get $a + bi = c + di$

The only way to make the equation is to make $ac + bd = 0$.

Injective

for every $a \cdot b$, there must be $a \in R$

Surjective

therefore prove by definition.
24. injective: Suppose \( g(y) = g(z) \)

\[
(y, z) = (y, z) \Rightarrow y = z.
\]

Not surjective: Let \( x \in M \) but there is no \( y \in R \) such that \( g(y) = (x) \).

By definition, a function \( g \) is injective if it is not surjective.

25. a) If \( g: R \to S \) and \( f: S \to T \) are homomorphisms, show that \( f \circ g \) is a homomorphism.

pf: \( f \circ g \) is a homomorphism.

pf: \( f \circ g((r + r)) = f(g(r + r)) = f(g(r) + g(r)) = f(g(r)) + f(g(r)) = f \circ g(r) + f \circ g(r) = f \circ g(r + r), \) so \( f \circ g \) is a homomorphism.

b) If \( f \) and \( g \) are isomorphisms, show \( f \circ g \) is an isomorphism.

pf: The composition of two injective/surjective maps is also injective/surjective. So, for \((\circ, \circ)\), assume \( f \circ g \) are injective/surjective.

\( f \) injective: Assume \( f(g(a)) = f(g(b)) \Rightarrow f(g(a)) = f(g(b)) \)

\[
\Rightarrow g(a) = g(b) \Rightarrow a = b.
\]

\( f \) surjective: Assume there is an element \( c \in T \) because \( f \) is surjective.

\( f(a) = c \) and because \( g \) is surjective \( g(c) = d \). Now, \( f(g(c)) = f(g(d)) = f(a) = c \). Therefore, \( f \circ g \) is surjective.
(b) This is preserved in $S$.

pf. Since $a$ is a zero divisor in $R$, $a \neq 0$ and $\exists b \in R$ s.t. $a \cdot b = 0$.

As $f$ is an isomorphism, $f$ is injective and surjective

$f(a) \neq f(b)$

then $f(a) \cdot f(b) = f(a) = 0$ in $S$.

Therefore $f(a)$ is a zero divisor in $S$.

(c) This is preserved.

pf. Since $a$ is a zero divisor in $R$.

As $f: R \rightarrow S$ is an isomorphism.

$f$ is an isomorphism

$f(a) = f(a) \cdot f(1) = f(a)$.

This is preserved.

pf. Since $R$ is an integral domain.

$R$ needs to be a commutative ring and $f: R \rightarrow S$.

As $f: R \rightarrow S$ is an isomorphism.

$f$ is injective and surjective

$f(a) = f(a)$

Therefore $f(a)$ is a zero divisor in $S$.

Therefore $S$ is a commutative ring with identity.
$\text{is an integral domain.}$

When ever $a \in D_{x}$ and $a \cdot b \in D_{x}$ then $a \cdot b \in D_{x}$

This also means

If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Since $a \cdot D_{x} = b \cdot D_{x}$

If $a \cdot 0 = (0 \cdot b) \in D_{x}$

Therefore $D_{x}$ is an integral domain.

\begin{align*}
\text{S4.1} & : & \text{V, W} \\
(a) & : & 3x^4 + [2 + y]x^3 + [-y + 1]x^2 + [1 + y]x + [4 + y] \\
& & 3x^4 + x^3 + 2x^2 + x \\
(b) & : & (x + 1)(x + 1) \cdot (x + 1) = (x^3 + 3x^2 + 3x + 1) \\
& & \cdot \frac{\sqrt{x^3 + 1}}{x} \\
(c) & : & (x - 1)^3 = (x - 1)^2 \cdot (x - 1) = (x - 1) \cdot (x^3 - 2x^2 + x - 1) \\
& & = (x^3 - 3x^2 + 3x - 1) \\
& & = x^3 - 3x^2 + 3x - 1 \\
(d) & : & (x^2 - 2x + 1)(2x^3 - 4x + 1) = 2x^5 - 4x^3 + 12x^2 - 6x + 12x^5 - 3x^4 + 3x^5 - 2x^5 + 1 \\
& & = 2x^5 + 3x^2 - 6x + 11x + 12 \\
& & = 2x^5 + 13x^2 - 6x + 11x + 12 \\
& & = 2x^5 + 13x^2 + 6x + 11x + 12
\end{align*}
4. a) ex. if \( f(x) = x^2 \) and \( g(x) = -x^2 \)
\[ f(x) + g(x) = x^2 - x^2 = 0, \quad \text{deg} = 0 \leq 2 \]

b) ex. if \( f(x) = x^3 \) and \( g(x) = 2x^2 \)
\[ \text{deg} [x^3 + 2x^2] = \max \text{deg} (x^3) , \text{deg} 2x^2 \]
\[ \Rightarrow 3 = 3 \]

4.1.4 a. Yes, closed under subtraction and multiplication.
b. No, not closed under multiplication.
c. No, not closed under multiplication.
d. Yes, closed under subtraction and multiplication.

e. No, not closed under multiplication.

Ex. \((a^2 + b)(x^3 + x) = x^6 + 2x^4 + x^2\)