Math 21C-2
Practice Final Exam
July 31, 2008

Name: Solutions

Signature: ____________________________

Student ID: __________________________

- There are ten (plus cover and bonus) pages to the exam.
- The exam totals 100 points, plus 10 bonus points.
- You will have 90 minutes to complete the exam.
- No calculators, notes, or books allowed.
- Good luck!

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1. (10 points) True or False. Remember, if a statement is not always true, then it is false. And if any part of the statement is false, the statement is false.

   a. (2 points) If \( a_n \geq 0 \) for all \( n \), \( \sum_{n=0}^{\infty} c_n \) converges, and \( a_n \geq c_n \) for all \( n \), then \( \sum_{n=0}^{\infty} a_n \) converges. \( \textit{False. Being bigger than terms that have a convergent sum tells you nothing.} \)

   b. (2 points) A power series can only converge at one of the endpoints of the interval of convergence. \( \textit{False. Consider } \sum_{n=0}^{\infty} \frac{x^n}{n^2} \text{ Interval of convergence is } [-1, 1]. \)

   c. (2 points) If a function \( f(x, y) \) has different limits along two different paths as \((x, y)\) approaches \((x_0, y_0)\), then \( \lim_{(x,y)\to(x_0,y_0)} f(x, y) \) does not exist. \( \textit{True.} \)

   d. (2 points) An absolute minimum of a function is also a local minimum. \( \textit{True.} \)

   e. (2 points) If \( f(x, y) = x^2 + y \), \( x(t) = \cos(t) \), \( y(t) = \sin(t) \), and \( t(s) = \frac{1}{s} \), then \( \frac{df}{ds} = -2\cos\left(\frac{1}{s}\right)\sin\left(\frac{1}{s}\right) + 2\cos\left(\frac{1}{s}\right) \).

\[ \textit{False.} \quad \frac{df}{ds} = \frac{df}{dx} \frac{dx}{ds} + \frac{df}{dy} \frac{dy}{dt} \frac{dt}{ds} \]
2. (10 points) Prove that if \( f(x) = \sum_{n=0}^{\infty} c_n x^n \), then \( c_n = \frac{f^{(n)}(0)}{n!} \) for all \( n \).

\[
\begin{align*}
  f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \\
  f'(0) &= c_1 \\
  f''(x) &= 2c_2 + 3c_3 x^2 + 4c_4 x^3 + \cdots \\
  f''(0) &= 2c_2 \\
  f'''(x) &= 3(2)c_3 x^2 + 4(3)c_4 x^3 + \cdots \\
  f'''(0) &= 3(2)c_3 \\
  \vdots \\
  f^{(n)}(0) &= n! c_n \\
\end{align*}
\]

So \( c_n = \frac{f^{(n)}(0)}{n!} \).
4. a. (10 points) A Hiker

a. (4 points) A hiker is walking straight up a hill that is at a 3° incline. She is \( h \) meters tall, and walking with speed \( v \) meters per second. A tenacious mosquito is flying horizontal circles of radius \( r \) meters about her head, the whole time. He is traveling at constant speed 2 meter per second in relation to her head. Write a vector-valued function to describe the position of the mosquito in space.

The position of the hiker's head can be described by
\[
\mathbf{r}(t) = h \mathbf{\hat{k}} + (v \cos 3^\circ) t \mathbf{\hat{j}} + (v \sin 3^\circ) t \mathbf{\hat{k}}
\]
\[
= (v \cos 3^\circ) t \mathbf{\hat{j}} + (v \sin 3^\circ) t + h \mathbf{\hat{k}}
\]

So the mosquito's position can be described by
\[
\mathbf{m}(t) = \mathbf{r}(t) + [(r \cos 2t) \mathbf{\hat{i}} + (r \sin 2t) \mathbf{\hat{j}}]
\]
\[
= (r \cos (2t)) \mathbf{\hat{i}} + ((v \cos 3^\circ) t + (r \sin (2t))) \mathbf{\hat{j}} + (v \sin 3^\circ) t + h \mathbf{\hat{k}}
\]

b. (3 points) What is the mosquito's velocity (with respect to the ground)?

\[
\frac{d\mathbf{m}}{dt} = (-2r \sin (2t)) \mathbf{\hat{i}} + (v \cos 3^\circ + 2r \cos (2t)) \mathbf{\hat{j}} + (v \sin 3^\circ) \mathbf{\hat{k}}
\]

c. (3 points) Find an equation for the line tangent to the mosquito's path at time \( t = \frac{\pi}{6} \).

\[
\left. \frac{d\mathbf{m}}{dt} \right|_{t=\frac{\pi}{6}} = (-2r \sin (\frac{\pi}{3})) \mathbf{\hat{i}} + (v \cos 3^\circ + 2r \cos (\frac{\pi}{3})) \mathbf{\hat{j}} + (v \sin 3^\circ) \mathbf{\hat{k}}
\]
\[
= (-r \sqrt{3}) \mathbf{\hat{i}} + (v \cos 3^\circ + r) \mathbf{\hat{j}} + (v \sin 3^\circ) \mathbf{\hat{k}}
\]

\[
\mathbf{m} \left( \frac{\pi}{6} \right) = (v \cos (\frac{\pi}{3})) \mathbf{\hat{i}} + ((v \cos 3^\circ) \frac{\pi}{6} + r \sin (\frac{\pi}{3})) \mathbf{\hat{j}} + ((v \sin 3^\circ) \frac{\pi}{6} + h) \mathbf{\hat{k}}
\]
\[
= \left( \frac{r \sqrt{3}}{2} \right) \mathbf{\hat{i}} + ((v \cos 3^\circ) \frac{\pi}{6} + \frac{\sqrt{3}}{2}) \mathbf{\hat{j}} + ((v \sin 3^\circ) \frac{\pi}{6} + h) \mathbf{\hat{k}}
\]

So the tangent line has equation:

\[
\mathbf{r} = \left( \frac{r \sqrt{3}}{2} \right) + t \left( -r \sqrt{3} \right)
\]
\[
\mathbf{j} = \left( (v \cos 3^\circ) \frac{\pi}{6} + \frac{\sqrt{3}}{2} \right) + t (v \cos 3^\circ + r)
\]
\[
\mathbf{k} = ((v \sin 3^\circ) \frac{\pi}{6} + h) + t (v \sin 3^\circ)
\]
5. (10 points) Find the following limits, if they exist. If the limit does not exist, explain why not.

a. (5 points) \( \lim_{(x,y) \to (2,2)} \frac{x + y - 4}{\sqrt{x + y} - 2} \).

\[
\lim_{(x,y) \to (2,2)} \frac{x + y - 4}{\sqrt{x + y} - 2} = \lim_{(x,y) \to (2,2)} \frac{x + y - 4}{\sqrt{x + y} - 2} \cdot \frac{\sqrt{x + y} + 2}{\sqrt{x + y} + 2} = \lim_{(x,y) \to (2,2)} \frac{(x + y - 4)(\sqrt{x + y} + 2)}{x + y - 4}
\]

\[
\lim_{(x,y) \to (2,2)} \sqrt{x + y} + 2 = \sqrt{2 + 2} + 2 = 2 + 2 = 4
\]

b. (5 points) \( \lim_{(x,y) \to (0,0)} \frac{|xy|}{xy} \).

The limit does not exist.

Consider the limit along \( \chi = \gamma \).

\[
\lim_{(x,y) \to (0,0)} \frac{|xy|}{xy} = \lim_{(x,y) \to (0,0)} \frac{|\chi \gamma|}{\chi \gamma} = 1
\]

But along \( \chi = -\gamma \).

\[
\lim_{(x,y) \to (0,0)} \frac{|xy|}{xy} = \lim_{(x,y) \to (0,0)} \frac{|\chi \gamma|}{-\chi \gamma} = -1
\]
6. (10 points) Use any method to find the following derivatives or partial derivatives.

a. (3 points) \( w = x^2 e^{2y} \cos(3z) \). Find \( \frac{dw}{dt} \) at \((1, \ln 2, 0)\), on the curve \( x = \cos(t), \ y = \ln(t+2), \ z = t \).

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

\[
=(2x e^{2x} \cos 3z) (-\sin t) + (2y^2 e^{2y} \cos (3z)) \left( \frac{1}{t+2} \right) + (-3x^2 e^{2x} \sin 3z) (1)
\]

\[
\frac{dw}{dt} \bigg|_{(1, \ln 2, 0)} = (2(1) e^{2 \ln 2} \cos (3 \cdot 0)) \left( -\sin (0) \right) + (2(1)^2 e^{2(1)} \cos (3\cdot 0)) \left( \frac{1}{1+2} \right) + (-3(1)^2 e^{2(1)} \sin (3 \cdot 0))
\]

\[
= 2(1) \left( \frac{1}{3} \right) + (-3)(0) = \frac{2}{3}
\]

b. (3 points) \( w = xy + \ln z, \ x = \frac{v^2}{u}, \ y = u + v, \ z = \cos u \). Find \( \frac{\partial w}{\partial v} \) when \((u, v) = (-1, 2)\).

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}
\]

\[
= (y) \left( \frac{2v}{u} \right) + (1) \left( \frac{1}{z} \right) \left( \frac{\partial z}{\partial v} \right)
\]

\[
= \frac{2v}{u} + \frac{1}{z} \left( \frac{\partial z}{\partial v} \right)
\]

\[
w = \frac{z^2}{u}, \quad \frac{\partial z}{\partial v} = -4, \quad y = (-1 + 2) = 1,
\]

\[
\frac{\partial w}{\partial v} \bigg|_{(u, v) = (-1, 2)} = (1) \left( \frac{2(1)}{-1} \right) + (-4) = \boxed{-8}
\]

c. (4 points) \( g(x, y) = x^2 y + \cos(y) + y \sin(x) \). Find \( \frac{\partial g}{\partial y} \).

\[
\frac{\partial g}{\partial y} = \frac{\partial}{\partial y} \left( \frac{2}{x} \right) \left( \frac{y}{x} \right) + \frac{\partial}{\partial y} \left( \frac{y}{x} \right) \left( x^2 \sin y + \cos y \sin x \right)
\]

\[
= \frac{2}{x} \left( \frac{y}{x} \right) - \frac{y}{x} \left( x^2 \sin y + \cos y \sin x \right)
\]

\[
= \boxed{0}
\]
7. (10 points) Superman has been trapped by the evil Lex Luthor. There is a block of Kryptonite at the coordinates $(0, 0, 0)$, creating an energy field with strength given by the function $K(x, y, z) = \left(\frac{1}{\text{distance from the source}}\right)$. What Lex Luthor does not realize is that there is also an intense yellow light source at $(4, 4, 0)$, with intensity given by $L(x, y, z) = \left(\frac{1}{\text{distance from the source}}\right)$. So Superman’s strength is given by $S(x, y, z) = L(x, y, z) - K(x, y, z)$. If Superman is at the coordinates $(5, 8, 2)$, then in what direction (unit vector) should Superman fly if he wants to increase his strength most rapidly?

We seek the direction of the gradient $\nabla S(5, 8, 2)$.

$$\nabla S = \langle S_x, S_y, S_z \rangle = \left\langle \frac{\partial}{\partial x} \left[ \frac{1}{(x-4)^2 + (y-4)^2 + z^2} \right], \frac{\partial}{\partial y} \left[ \frac{1}{(x-4)^2 + (y-4)^2 + z^2} \right], \frac{\partial}{\partial z} \left[ \frac{1}{(x-4)^2 + (y-4)^2 + z^2} \right] \right\rangle$$

$$= \left\langle \frac{-2(x - 4)}{2((x-4)^2 + (y-4)^2 + z^2)^{3/2}}, \frac{2y}{2((x-4)^2 + (y-4)^2 + z^2)^{3/2}}, \frac{-2(z - 4)}{2((x-4)^2 + (y-4)^2 + z^2)^{3/2}} \right\rangle$$

$$= \left\langle \frac{-2}{2((5-4)^2 + (8-4)^2 + 2^2)^{3/2}}, \frac{10}{25^2}, \frac{16}{25^2}, 0 \right\rangle$$

And clearly it is unreasonable to calculate:

$$\nabla S \big|_{(5, 8, 2)}$$

$$\left| \nabla S \big|_{(5, 8, 2)} \right|$$
8. (10 points) Find two numbers $a$ and $b$ with $a \leq b$ such that

$$\int_a^b (6 - x + x^2) \, dx$$

has its largest value.

Let $f(a, b) = \int_a^b (6 - x + x^2) \, dx$. We want the maximal value of $f$

over the domain $\{(a, b) \mid a \leq b\}$

$$f(a, b) = \int_a^b (6 - x + x^2) \, dx = \left[6x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right]_a^b$$

$$= \left(6b - \frac{1}{2}b^2 + \frac{1}{3}b^3\right) - \left(6a - \frac{1}{2}a^2 + \frac{1}{3}a^3\right)$$

$$\frac{df}{da} = -6 + a - a^2 , \quad \frac{df}{db} = 6 - b + b^2$$

exist everywhere.

So the only critical points are where $\frac{df}{da} = \frac{df}{db} = 0$.

$$-6 + a - a^2 = 0 \Rightarrow \text{No solution}.$$ 

There are no critical values on the interior of this domain.

At every point on the boundary, $a = b$, $f(a, a) = 0$.

So there is no choice for $a, b$ which maximize $\int_a^b (6 - x + x^2) \, dx$. 
9. (10 points) Find all the relative maxima, relative minima, and saddle points of the function:

\[ f(x, y) = \frac{1}{x} + xy + \frac{1}{y} \]

\[ \frac{\partial f}{\partial x} = -\frac{1}{x^2} + y \quad \frac{\partial f}{\partial y} = x - \frac{1}{y^2} \]

Exist as long as \( x \neq 0, \ y \neq 0 \).

But these points are not in the domain of the function anyway.

So the only critical points in the domain are where

\[ \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \]

\[-\frac{1}{x^2} + y = 0 \quad \Rightarrow \quad y = \frac{1}{x^2} \]

\[ x - \frac{1}{y^2} = 0 \quad \Rightarrow \quad x = \frac{1}{(x^2)^2} = 0 \]

\[ x - y = 0 \quad x(1 - x^2) = 0 \]

So \( x = 0 \) or \( x = 1 \)

But \( x \) cannot be 0 (not in domain).

So \( x = 1 \) \( \Rightarrow \) \( y = \frac{1}{1^2} = 1 \)

So the only critical pt. is \((1, 1)\).

\[ \frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3} \quad \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3} \quad \frac{\partial^2 f}{\partial x \partial y} = 1 \]

At \((1, 1)\),

\[ \left( \frac{\partial^2 f}{\partial x^2} \right) \left( \frac{\partial^2 f}{\partial y^2} \right) - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 = \left( \frac{2}{1^3} \right) \left( \frac{2}{1^3} \right) - (1)^2 = 4 - 1 = 3 > 0 \]

And \( \frac{\partial^2 f}{\partial x^2} = \frac{2}{1^3} = 2 > 0 \),

So \( f \) has a Local Minimum at \((1, 1)\) of

\[ f(1, 1) = \frac{1}{1} + 1(1)(1) + \frac{1}{1} = 3 \]
10. (10 points) Maximize the function \( f(x, y, z) = x^2 + 2y - z^2 \) subject to the constraints \( 2x - y = 0 \) and \( y + z = 0 \).

\[
\begin{align*}
\alpha_1(x, y, z) &= 2x - y, & \alpha_2(x, y, z) &= y + z \\
\n\n\end{align*}
\]

We want \( x, y, z, \lambda, \mu \) so that

\[
\nabla f = \lambda \nabla \alpha_1 + \mu \nabla \alpha_2, & \quad \alpha_1 = 0, \quad \alpha_2 = 0 \\
\Rightarrow < 2x, 2, -2z > = \lambda < 2, -1, 0 > + \mu < 0, 1, 1 >
\]

So we want

\[
\begin{align*}
2x &= 2\lambda, & 2 &= -\lambda + \mu, & -2z &= \mu, \\
2x - y &= 0, & \text{and} & y + z &= 0.
\end{align*}
\]

\[
\begin{align*}
2x &= 2\lambda \Rightarrow x = \lambda, & \text{and} & -2z &= \mu
\end{align*}
\]

\[
\begin{align*}
2 &= -\lambda + \mu \Rightarrow 2 &= -x + (-2z) \\
x &= -2 - 2z
\end{align*}
\]

Then, \( 2 (-2z) - y = 0 \Rightarrow y = -4 - 4z \),

\[
\begin{align*}
(-4 - 4z) + z &= 0 \Rightarrow -4 = 3z \Rightarrow \\
\lambda &= \frac{-4}{3}, \quad \mu = \frac{4}{3}
\end{align*}
\]

And \( x = -2 - 2(-\frac{4}{3}) = \frac{-6}{3} + \frac{8}{3} = \frac{2}{3} = \lambda \)

So the only critical point is \((\frac{2}{3}, \frac{4}{3}, -\frac{4}{3})\),

\[
\begin{align*}
\text{Where,} & \quad f(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}) = (\frac{2}{3})^3 + 2(\frac{4}{3}) - (-\frac{4}{3})^2 = \frac{4}{9} + \frac{8}{3} - \frac{16}{9} = \frac{4}{3}
\end{align*}
\]

Clearly \((0,0,0)\) is part of our constrained domain, and \( f(0,0,0) = 0 \), so \( \frac{4}{3} \) must be a maximum.
**Bonus.** (10 points) Find a solution to the initial value problem

\[ y' - y = x \quad y(0) = 1 \]

by assuming that there is a solution of the form

\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]

\[ y' = a_1 + 2 a_2 x + 3 a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots \]

If this satisfies \( y' - y = x \), then

\[ a_1 - a_0 = 0 \]
\[ 2a_2 - a_1 = 1 \]
\[ 3a_3 - a_2 = 0 \]
\[ \vdots \]
\[ n a_n - a_{n-1} = 0 \]

Further, \( y(0) = 1 \), so

\[ a_0 + a_1 (0) + a_2 (0)^2 + \cdots = 1 \]

\[ \Rightarrow a_0 = 1 \]

So \( a_1 = 1 \), \( 2a_2 - 1 = 1 \) \( \Rightarrow \ a_2 = \frac{2}{2} = 1 \)

\[ 3a_3 - a_2 = 0 \] \( \Rightarrow \ a_3 = \frac{1}{3} \)

\[ 4a_4 - a_3 = 0 \] \( \Rightarrow \ a_4 = \frac{1}{3 \cdot 4} \)

\[ \vdots \]

\[ n a_n - a_{n-1} = 0 \] \( \Rightarrow \ a_n = \frac{1}{n!} \)

\[ y = 1 + x + 2 \frac{x^2}{2!} + 2 \frac{x^3}{3!} + 2 \frac{x^4}{4!} + \cdots \]

\[ = 1 + x + 2 \left( e^x - 1 - x \right)_{12} \]

\[ y' = 2 e^x - x - 1 \]