Lagrange Polynomial Approximation

## 4.3 Lagrange Approximation

Interpolation means to estimate a missing function value by taking a weighted average of known function values at neighboring points. Linear interpolation uses a line segment that passes through two points. The slope between $(x_0, y_0)$ and $(x_1, y_1)$ is $m = (y_1 - y_0)/(x_1 - x_0)$, and the point-slope formula for the line $y = m(x - x_0) + y_0$ can be rearranged as

$$y = P(x) = y_0 + (y_1 - y_0)\frac{x - x_0}{x_1 - x_0}. \tag{1}$$

When formula (1) is expanded, the result is a polynomial of degree $\le 1$. Evaluation of $P(x)$ at $x_0$ and $x_1$ produces $y_0$ and $y_1$, respectively:

$$\begin{aligned} P(x_0) &= y_0 + (y_1 - y_0)(0) = y_0, \\ P(x_1) &= y_0 + (y_1 - y_0)(1) = y_1. \end{aligned} \tag{2}$$

The French mathematician Joseph Louis Lagrange used a slightly different method to find this polynomial. He noticed that it could be written as

$$y = P_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}. \tag{3}$$

Each term on the right side of (3) involves a linear factor; hence the sum is a polynomial of degree $\le 1$. The quotients in (3) are denoted by

$$L_{1,0}(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_{1,1}(x) = \frac{x - x_0}{x_1 - x_0}. \tag{4}$$

Computation reveals that $L_{1,0}(x_0) = 1$, $L_{1,0}(x_1) = 0$, $L_{1,1}(x_0) = 0$, and $L_{1,1}(x_1) = 1$ so that the polynomial $P_1(x)$ in (3) also passes through the two given points:

$$P_1(x_0) = y_0 + y_1(0) = y_0 \quad \text{and} \quad P_1(x_1) = y_0(0) + y_1 = y_1. \tag{5}$$

The terms $L_{1,0}(x)$ and $L_{1,1}(x)$ in (4) are called ***Lagrange coefficient polynomials*** based on the nodes $x_0$ and $x_1$. Using this notation, (3) can be written in summation form

$$P_1(x) = \sum_{k=0}^{1} y_k L_{1,k}(x). \tag{6}$$

Suppose that the ordinates $y_k$ are computed with the formula $y_k = f(x_k)$. If $P_1(x)$ is used to approximate $f(x)$ over the interval $[x_0, x_1]$, we call the process ***interpolation***. If $x < x_0$ (or $x_1 < x$), then using $P_1(x)$ is called ***extrapolation***. The next example illustrates these concepts.

**Example 4.6.** Consider the graph $y = f(x) = \cos(x)$ over $[0.0, 1.2]$.

**(a)** Use the nodes $x_0 = 0.0$ and $x_1 = 1.2$ to construct a linear interpolation polynomial $P_1(x)$.

**(b)** Use the nodes $x_0 = 0.2$ and $x_1 = 1.0$ to construct a linear approximating polynomial $Q_1(x)$.

**(a)** Using (3) with the abscissas $x_0 = 0.0$ and $x_1 = 1.2$ and the ordinates $y_0 = \cos(0.0) = 1.000000$ and $y_1 = \cos(1.2) = 0.362358$ produces

$$\begin{aligned} P_1(x) &= 1.000000 \frac{x - 1.2}{0.0 - 1.2} + 0.362358 \frac{x - 0.0}{1.2 - 0.0} \\ &= -0.833333(x - 1.2) + 0.301965(x - 0.0). \end{aligned}$$

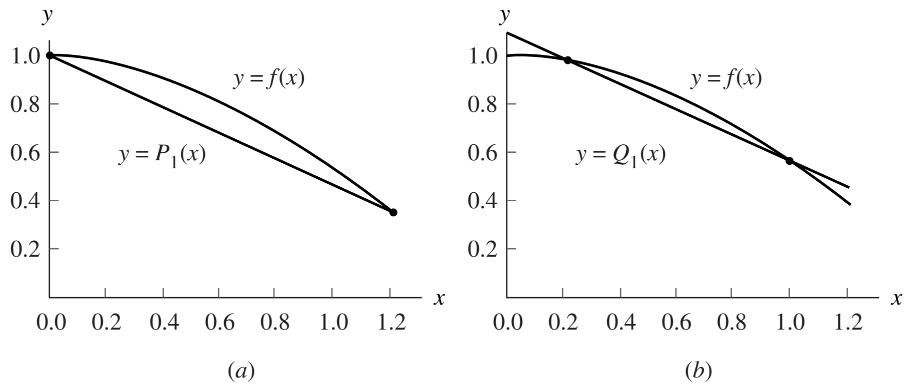


**Figure 4.11** (a) The linear approximation $y = P_1(x)$ where the nodes $x_0 = 0.0$ and $x_1 = 1.2$ are the endpoints of the interval $[a, b]$. (b) The linear approximation $y = Q_1(x)$ where the nodes $x_0 = 0.2$ and $x_1 = 1.0$ lie inside the interval $[a, b]$.

**(b)** When the nodes $x_0 = 0.2$ and $x_1 = 1.0$ with $y_0 = \cos(0.2) = 0.980067$ and $y_1 = \cos(1.0) = 0.540302$ are used, the result is

$$\begin{aligned} Q_1(x) &= 0.980067\frac{x - 1.0}{0.2 - 1.0} + 0.540302\frac{x - 0.2}{1.0 - 0.2} \\ &= -1.225083(x - 1.0) + 0.675378(x - 0.2). \end{aligned}$$

Figure 4.11(a) and (b) show the graph of $y = \cos(x)$ and compare it with $y = P_1(x)$ and $y = Q_1(x)$, respectively. Numerical computations are given in Table 4.6 and reveal that $Q_1(x)$ has less error at the points $x_k$ that satisfy $0.1 \le x_k \le 1.1$. The largest tabulated error, $f(0.6) - P_1(0.6) = 0.144157$, is reduced to $f(0.6) - Q_1(0.6) = 0.065151$ by using $Q_1(x)$. ■

The generalization of (6) is the construction of a polynomial $P_N(x)$ of degree at most $N$ that passes through the $N + 1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)$ and has the form

$$P_N(x) = \sum_{k=0}^{N} y_k L_{N,k}(x), \tag{7}$$

where $L_{N,k}$ is the Lagrange coefficient polynomial based on these nodes:

$$L_{N,k}(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_N)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_N)}. \tag{8}$$

It is understood that the terms $(x - x_k)$ and $(x_k - x_k)$ do not appear on the right side of

**Table 4.6** Comparison of $f(x) = \cos(x)$ and the Linear Approximations $P_1(x)$ and $Q_1(x)$

| $x_k$ | $f(x_k) = \cos(x_k)$ | $P_1(x_k)$ | $f(x_k) - P_1(x_k)$ | $Q_1(x_k)$ | $f(x_k) - Q_1(x_k)$ |
|---|---|---|---|---|---|
| 0.0 | 1.000000 | 1.000000 | 0.000000 | 1.090008 | −0.090008 |
| 0.1 | 0.995004 | 0.946863 | 0.048141 | 1.035037 | −0.040033 |
| 0.2 | 0.980067 | 0.893726 | 0.086340 | 0.980067 | 0.000000 |
| 0.3 | 0.955336 | 0.840589 | 0.114747 | 0.925096 | 0.030240 |
| 0.4 | 0.921061 | 0.787453 | 0.133608 | 0.870126 | 0.050935 |
| 0.5 | 0.877583 | 0.734316 | 0.143267 | 0.815155 | 0.062428 |
| 0.6 | 0.825336 | 0.681179 | 0.144157 | 0.760184 | 0.065151 |
| 0.7 | 0.764842 | 0.628042 | 0.136800 | 0.705214 | 0.059628 |
| 0.8 | 0.696707 | 0.574905 | 0.121802 | 0.650243 | 0.046463 |
| 0.9 | 0.621610 | 0.521768 | 0.099842 | 0.595273 | 0.026337 |
| 1.0 | 0.540302 | 0.468631 | 0.071671 | 0.540302 | 0.000000 |
| 1.1 | 0.453596 | 0.415495 | 0.038102 | 0.485332 | −0.031736 |
| 1.2 | 0.362358 | 0.362358 | 0.000000 | 0.430361 | −0.068003 |

equation (8). It is appropriate to introduce the product notation for (8), and we write

$$L_{N,k}(x) = \frac{\prod_{\substack{j=0 \\ j\neq k}}^{N}(x - x_j)}{\prod_{\substack{j=0 \\ j\neq k}}^{N}(x_k - x_j)}. \tag{9}$$

Here the notation in (9) indicates that in the numerator the product of the linear factors $(x - x_j)$ is to be formed, but the factor $(x - x_k)$ is to be left out (or skipped). A similar construction occurs in the denominator.

A straightforward calculation shows that for each fixed $k$, the Lagrange coefficient polynomial $L_{N,k}(x)$ has the property

$$L_{N,k}(x_j) = 1 \ \text{when} \ j = k \qquad \text{and} \qquad L_{N,k}(x_j) = 0 \ \text{when} \ j \neq k. \tag{10}$$

Then direct substitution of these values into (7) is used to show that the polynomial curve $y = P_N(x)$ goes through $(x_j, y_j)$:

$$\begin{aligned} P_N(x_j) &= y_0 L_{N,0}(x_j) + \cdots + y_j L_{N,j}(x_j) + \cdots + y_N L_{N,N}(x_j) \\ &= y_0(0) + \cdots + y_j(1) + \cdots + y_N(0) = y_j. \end{aligned} \tag{11}$$

To show that $P_N(x)$ is unique, we invoke the fundamental theorem of algebra, which states that a nonzero polynomial $T(x)$ of degree $\leq N$ has at most $N$ roots. In other words, if $T(x)$ is zero at $N + 1$ distinct abscissas, it is identically zero. Suppose that $P_N(x)$ is not unique and that there exists another polynomial $Q_N(x)$ of degree $\leq N$ that also passes through the $N+1$ points. Form the difference polynomial $T(x) =$

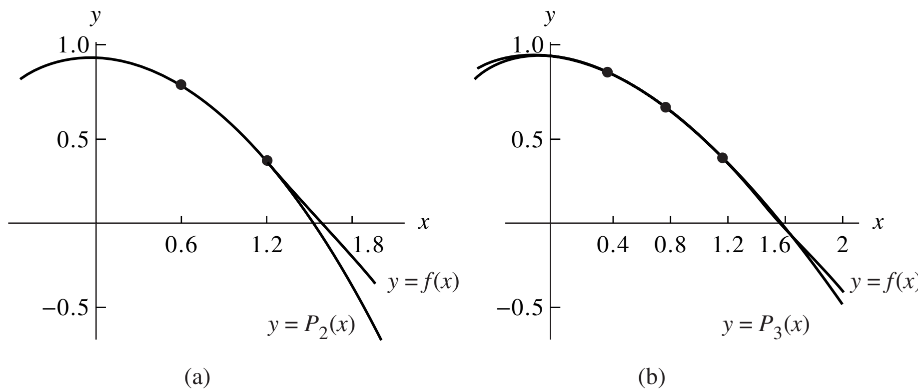


**Figure 4.12** (a) The quadratic approximation polynomial $y = P_2(x)$ based on the nodes $x_0 = 0.0$, $x_1 = 0.6$, and $x_2 = 1.2$. (b) The cubic approximation polynomial $y = P_3(x)$ based on the nodes $x_0 = 0.0$, $x_1 = 0.4$, $x_2 = 0.8$, and $x_3 = 1.2$.

$P_N(x) - Q_N(x)$. Observe that the polynomial $T(x)$ has degree $\leq N$ and that $T(x_j) = P_N(x_j) - Q_N(x_j) = y_j - y_j = 0$, for $j = 0, 1, \ldots, N$. Therefore, $T(x) \equiv 0$ and it follows that $Q_N(x) = P_N(x)$.

When (7) is expanded, the result is similar to (3). The Lagrange quadratic interpolating polynomial through the three points $(x_0, y_0)$, $(x_1, y_1)$, and $(x_2, y_2)$ is

$$(12) \quad P_2(x) = y_0\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1\frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

The Lagrange cubic interpolating polynomial through the four points $(x_0, y_0)$, $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ is

$$(13) \quad \begin{aligned} P_3(x) = {} & y_0\frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + y_1\frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\ & + y_2\frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + y_3\frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}. \end{aligned}$$

**Example 4.7.** Consider $y = f(x) = \cos(x)$ over $[0.0, 1.2]$.

**(a)** Use the three nodes $x_0 = 0.0, x_1 = 0.6$, and $x_2 = 1.2$ to construct a quadratic interpolation polynomial $P_2(x)$.

**(b)** Use the four nodes $x_0 = 0.0$, $x_1 = 0.4$, $x_2 = 0.8$, and $x_3 = 1.2$ to construct a cubic interpolation polynomial $P_3(x)$.

**(a)** Using $x_0 = 0.0, x_1 = 0.6$, $x_2 = 1.2$ and $y_0 = \cos(0.0) = 1, y_1 = \cos(0.6) =$

0.825336, and $y_2 = \cos(1.2) = 0.362358$ in equation (12) produces

$$\begin{aligned}P_2(x) &= 1.0\frac{(x-0.6)(x-1.2)}{(0.0-0.6)(0.0-1.2)} + 0.825336\frac{(x-0.0)(x-1.2)}{(0.6-0.0)(0.6-1.2)}\\ &\quad + 0.362358\frac{(x-0.0)(x-0.6)}{(1.2-0.0)(1.2-0.6)}\\ &= 1.388889(x-0.6)(x-1.2) - 2.292599(x-0.0)(x-1.2)\\ &\quad + 0.503275(x-0.0)(x-0.6).\end{aligned}$$

**(b)** Using $x_0 = 0.0$, $x_1 = 0.4$, $x_2 = 0.8$, $x_3 = 1.2$ and $y_0 = \cos(0.0) = 1.0$, $y_1 = \cos(0.4) = 0.921061$, $y_2 = \cos(0.8) = 0.696707$, and $y_3 = \cos(1.2) = 0.362358$ in equation (13) produces

$$\begin{aligned}P_3(x) &= 1.000000\frac{(x-0.4)(x-0.8)(x-1.2)}{(0.0-0.4)(0.0-0.8)(0.0-1.2)}\\ &\quad + 0.921061\frac{(x-0.0)(x-0.8)(x-1.2)}{(0.4-0.0)(0.4-0.8)(0.4-1.2)}\\ &\quad + 0.696707\frac{(x-0.0)(x-0.4)(x-1.2)}{(0.8-0.0)(0.8-0.4)(0.8-1.2)}\\ &\quad + 0.362358\frac{(x-0.0)(x-0.4)(x-0.8)}{(1.2-0.0)(1.2-0.4)(1.2-0.8)}\\ &= -2.604167(x-0.4)(x-0.8)(x-1.2)\\ &\quad + 7.195789(x-0.0)(x-0.8)(x-1.2)\\ &\quad - 5.443021(x-0.0)(x-0.4)(x-1.2)\\ &\quad + 0.943641(x-0.0)(x-0.4)(x-0.8).\end{aligned}$$

The graphs of $y = \cos(x)$ and the polynomials $y = P_2(x)$ and $y = P_3(x)$ are shown in Figure 4.12(a) and (b), respectively. ■

## Error Terms and Error Bounds

It is important to understand the nature of the error term when the Lagrange polynomial is used to approximate a continuous function $f(x)$. It is similar to the error term for the Taylor polynomial, except that the factor $(x - x_0)^{N+1}$ is replaced with the product $(x - x_0)(x - x_1)\cdots(x - x_N)$. This is expected because interpolation is exact at each of the $N + 1$ nodes $x_k$, where we have $E_N(x_k) = f(x_k) - P_N(x_k) = y_k - y_k = 0$ for $k = 0, 1, 2, \ldots, N$.

**Theorem 4.3 (Lagrange Polynomial Approximation).** Assume that $f \in C^{N+1}[a, b]$ and that $x_0, x_1, \ldots, x_N \in [a, b]$ are $N + 1$ nodes. If $x \in [a, b]$, then

(14) $$f(x) = P_N(x) + E_N(x),$$

where $P_N(x)$ is a polynomial that can be used to approximate $f(x)$:

$$f(x) \approx P_N(x) = \sum_{k=0}^{N} f(x_k) L_{N,k}(x). \tag{15}$$

The error term $E_N(x)$ has the form

$$E_N(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_N) f^{(N+1)}(c)}{(N+1)!} \tag{16}$$

for some value $c = c(x)$ that lies in the interval $[a, b]$.

*Proof.* As an example of the general method, we establish (16) when $N = 1$. The general case is discussed in the exercises. Start by defining the special function $g(t)$ as follows:

$$g(t) = f(t) - P_1(t) - E_1(x) \frac{(t - x_0)(t - x_1)}{(x - x_0)(x - x_1)}. \tag{17}$$

Notice that $x$, $x_0$, and $x_1$ are constants with respect to the variable $t$ and that $g(t)$ evaluates to be zero at these three values; that is,

$$\begin{aligned}
g(x) &= f(x) - P_1(x) - E_1(x) \frac{(x - x_0)(x - x_1)}{(x - x_0)(x - x_1)} = f(x) - P_1(x) - E_1(x) = 0, \\
g(x_0) &= f(x_0) - P_1(x_0) - E_1(x) \frac{(x_0 - x_0)(x_0 - x_1)}{(x - x_0)(x - x_1)} = f(x_0) - P_1(x_0) = 0, \\
g(x_1) &= f(x_1) - P_1(x_1) - E_1(x) \frac{(x_1 - x_0)(x_1 - x_1)}{(x - x_0)(x - x_1)} = f(x_1) - P_1(x_1) = 0.
\end{aligned}$$

Suppose that $x$ lies in the open interval $(x_0, x_1)$. Applying Rolle's theorem to $g(t)$ on the interval $[x_0, x]$ produces a value $d_0$, with $x_0 < d_0 < x$, such that

$$g'(d_0) = 0. \tag{18}$$

A second application of Rolle's theorem to $g(t)$ on $[x, x_1]$ will produce a value $d_1$, with $x < d_1 < x_1$, such that

$$g'(d_1) = 0. \tag{19}$$

Equations (18) and (19) show that the function $g'(t)$ is zero at $t = d_0$ and $t = d_1$. A third use of Rolle's theorem, but this time applied to $g'(t)$ over $[d_0, d_1]$, produces a value $c$ for which

$$g^{(2)}(c) = 0. \tag{20}$$

Now go back to (17) and compute the derivatives $g'(t)$ and $g''(t)$:

$$g'(t) = f'(t) - P_1'(t) - E_1(x)\frac{(t-x_0)+(t-x_1)}{(x-x_0)(x-x_1)}, \tag{21}$$

$$g''(t) = f''(t) - 0 - E_1(x)\frac{2}{(x-x_0)(x-x_1)}. \tag{22}$$

In (22) we have used the fact the $P_1(t)$ is a polynomial of degree $N = 1$; hence its second derivative is $P_1''(t) \equiv 0$. Evaluation of (22) at the point $t = c$ and using (20) yields

$$0 = f''(c) - E_1(x)\frac{2}{(x-x_0)(x-x_1)}. \tag{23}$$

Solving (23) for $E_1(x)$ results in the desired form (16) for the remainder:

$$E_1(x) = \frac{(x-x_0)(x-x_1)f^{(2)}(c)}{2!}, \tag{24}$$

and the proof is complete. •

The next result addresses the special case when the nodes for the Lagrange polynomial are equally spaced $x_k = x_0 + hk$, for $k = 0, 1, \ldots, N$, and the polynomial $P_N(x)$ is used only for interpolation inside the interval $[x_0, x_N]$.

**Theorem 4.4 (Error Bounds for Lagrange Interpolation, Equally Spaced Nodes).** Assume that $f(x)$ is defined on $[a, b]$, which contains equally spaced nodes $x_k = x_0 + hk$. Additionally, assume that $f(x)$ and the derivatives of $f(x)$, up to the order $N + 1$, are continuous and bounded on the special subintervals $[x_0, x_1]$, $[x_0, x_2]$, and $[x_0, x_3]$, respectively; that is,

$$|f^{(N+1)}(x)| \le M_{N+1} \quad \text{for } x_0 \le x \le x_N, \tag{25}$$

for $N = 1, 2, 3$. The error terms (16) corresponding to the cases $N = 1, 2$, and 3 have the following useful bounds on their magnitude:

$$|E_1(x)| \le \frac{h^2 M_2}{8} \quad \text{valid for } x \in [x_0, x_1], \tag{26}$$

$$|E_2(x)| \le \frac{h^3 M_3}{9\sqrt{3}} \quad \text{valid for } x \in [x_0, x_2], \tag{27}$$

$$|E_3(x)| \le \frac{h^4 M_4}{24} \quad \text{valid for } x \in [x_0, x_3]. \tag{28}$$

*Proof.* We establish (26) and leave the others for the reader. Using the change of variables $x - x_0 = t$ and $x - x_1 = t - h$, the error term $E_1(x)$ can be written as

$$E_1(x) = E_1(x_0 + t) = \frac{(t^2 - ht)f^{(2)}(c)}{2!} \quad \text{for } 0 \le t \le h. \tag{29}$$

The bound for the derivative for this case is

$$(30) \qquad |f^{(2)}(c)| \le M_2 \quad \text{for } x_0 \le c \le x_1.$$

Now determine a bound for the expression $(t^2 - ht)$ in the numerator of (29); call this term $\Phi(t) = t^2 - ht$. Since $\Phi'(t) = 2t - h$, there is one critical point $t = h/2$ that is the solution to $\Phi'(t) = 0$. The extreme values of $\Phi(t)$ over $[0, h]$ occur either at an end point $\Phi(0) = 0$, $\Phi(h) = 0$ or at the critical point $\Phi(h/2) = -h^2/4$. Since the latter value is the largest, we have established the bound

$$(31) \qquad |\Phi(t)| = |t^2 - ht| \le \frac{|-h^2|}{4} = \frac{h^2}{4} \quad \text{for } 0 \le t \le h.$$

Using (30) and (31) to estimate the magnitude of the product in the numerator in (29) results in

$$(32) \qquad |E_1(x)| = \frac{|\Phi(t)||f^{(2)}(c)|}{2!} \le \frac{h^2 M_2}{8},$$

and formula (26) is established. •

## Comparison of Accuracy and $O(h^{N+1})$

The significance of Theorem 4.4 is to understand a simple relationship between the size of the error terms for linear, quadratic, and cubic interpolation. In each case the error bound $|E_N(x)|$ depends on $h$ in two ways. First, $h^{N+1}$ is explicitly present so that $|E_N(x)|$ is proportional to $h^{N+1}$. Second, the values $M_{N+1}$ generally depend on $h$ and tend to $|f^{(N+1)}(x_0)|$ as $h$ goes to zero. Therefore, as $h$ goes to zero, $|E_N(x)|$ converges to zero with the same rapidity that $h^{N+1}$ converges to zero. The notation $\boldsymbol{O}(h^{N+1})$ is used when discussing this behavior. For example, the error bound (26) can be expressed as

$$|E_1(x)| = \boldsymbol{O}(h^2) \quad \text{valid for } x \in [x_0, x_1].$$

The notation $\boldsymbol{O}(h^2)$ stands in place of $h^2M_2/8$ in relation (26) and is meant to convey the idea that the bound for the error term is approximately a multiple of $h^2$; that is,

$$|E_1(x)| \le Ch^2 \approx \boldsymbol{O}(h^2).$$

As a consequence, if the derivatives of $f(x)$ are uniformly bounded on the interval $[a, b]$ and $|h| < 1$, then choosing $N$ large will make $h^{N+1}$ small, and the higher-degree approximating polynomial will have less error.

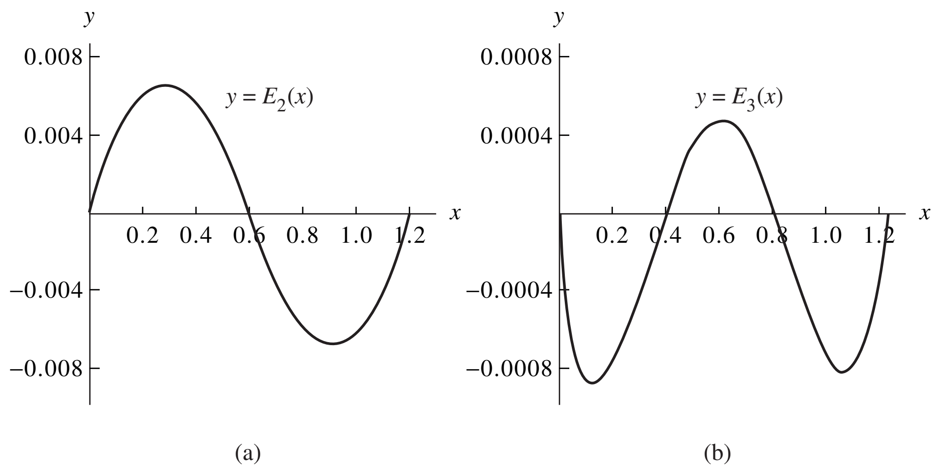


**Figure 4.13** (a) The error function $E_2(x) = \cos(x) - P_2(x)$. (b) The error function $E_3(x) = \cos(x) - P_3(x)$.

**Example 4.8.** Consider $y = f(x) = \cos(x)$ over [0.0, 1.2]. Use formulas (26) through (28) and determine the error bounds for the Lagrange polynomials $P_1(x)$, $P_2(x)$, and $P_3(x)$ that were constructed in Examples 4.6 and 4.7.

First, determine the bounds $M_2$, $M_3$, and $M_4$ for the derivatives $|f^{(2)}(x)|$, $|f^{(3)}(x)|$, and $|f^{(4)}(x)|$, respectively, taken over the interval [0.0, 1.2]:

$$|f^{(2)}(x)| = |-\cos(x)| \le |-\cos(0.0)| = 1.000000 = M_2,$$
$$|f^{(3)}(x)| = |\sin(x)| \le |\sin(1.2)| = 0.932039 = M_3,$$
$$|f^{(4)}(x)| = |\cos(x)| \le |\cos(0.0)| = 1.000000 = M_4.$$

For $P_1(x)$ the spacing of the nodes is $h = 1.2$, and its error bound is

$$|E_1(x)| \le \frac{h^2 M_2}{8} \le \frac{(1.2)^2(1.000000)}{8} = 0.180000. \tag{33}$$

For $P_2(x)$ the spacing of the nodes is $h = 0.6$, and its error bound is

$$|E_2(x)| \le \frac{h^3 M_3}{9\sqrt{3}} \le \frac{(0.6)^3(0.932039)}{9\sqrt{3}} = 0.012915. \tag{34}$$

For $P_3(x)$ the spacing of the nodes is $h = 0.4$, and its error bound is

$$|E_3(x)| \le \frac{h^4 M_4}{24} \le \frac{(0.4)^4(1.000000)}{24} = 0.001067. \tag{35}$$ ■

From Example 4.6 we saw that $|E_1(0.6)| = |\cos(0.6) - P_1(0.6)| = 0.144157$, so the bound 0.180000 in (33) is reasonable. The graphs of the error functions $E_2(x) = \cos(x) - P_2(x)$ and $E_3(x) = \cos(x) - P_3(x)$ are shown in Figure 4.13(a) and (b),

**Table 4.7** Comparison of $f(x) = \cos(x)$ and the Quadratic and Cubic Polynomial Approximations $P_2(x)$ and $P_3(x)$

| $x_k$ | $f(x_k) = \cos(x_k)$ | $P_2(x_k)$ | $E_2(x_k)$ | $P_3(x_k)$ | $E_3(x_k)$ |
|---|---|---|---|---|---|
| 0.0 | 1.000000 | 1.000000 | 0.0 | 1.000000 | 0.0 |
| 0.1 | 0.995004 | 0.990911 | 0.004093 | 0.995835 | −0.000831 |
| 0.2 | 0.980067 | 0.973813 | 0.006253 | 0.980921 | −0.000855 |
| 0.3 | 0.955336 | 0.948707 | 0.006629 | 0.955812 | −0.000476 |
| 0.4 | 0.921061 | 0.915592 | 0.005469 | 0.921061 | 0.0 |
| 0.5 | 0.877583 | 0.874468 | 0.003114 | 0.877221 | 0.000361 |
| 0.6 | 0.825336 | 0.825336 | 0.0 | 0.824847 | 0.00089 |
| 0.7 | 0.764842 | 0.768194 | −0.003352 | 0.764491 | 0.000351 |
| 0.8 | 0.696707 | 0.703044 | −0.006338 | 0.696707 | 0.0 |
| 0.9 | 0.621610 | 0.629886 | −0.008276 | 0.622048 | −0.000438 |
| 1.0 | 0.540302 | 0.548719 | −0.008416 | 0.541068 | −0.000765 |
| 1.1 | 0.453596 | 0.459542 | −0.005946 | 0.454320 | −0.000724 |
| 1.2 | 0.362358 | 0.362358 | 0.0 | 0.362358 | 0.0 |

respectively, and numerical computations are given in Table 4.7. Using values in the table, we find that $|E_2(1.0)| = |\cos(1.0) - P_2(1.0)| = 0.008416$ and $|E_3(0.2)| = |\cos(0.2) - P_3(0.2)| = 0.000855$, which is in reasonable agreement with the bounds 0.012915 and 0.001607 given in (34) and (35), respectively.

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