7.5 Gauss-Legendre Integration

We wish to find the area under the curve

$$y = f(x), \qquad -1 \le x \le 1.$$

What method gives the best answer if only two function evaluations are to be made? We have already seen that the trapezoidal rule is a method for finding the area under the curve and that it uses two function evaluations at the end points $(-1, f(-1))$, and $(1, f(1))$. But if the graph of $y = f(x)$ is concave down, the error in approximation is the entire region that lies between the curve and the line segment joining the points; another instance is shown in Figure 7.10(a).

If we can use nodes x_1 and x_2 that lie inside the interval $[-1, 1]$, the line through the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ crosses the curve, and the area under the line more closely approximates the area under the curve (see Figure 7.10(b)). The equation of the line is

$$y = f(x_1) + \frac{(x - x_1)(f(x_2) - f(x_1))}{x_2 - x_1} \tag{1}$$

and the area of the trapezoid under the line is

$$A_{\text{trap}} = \frac{2x_2}{x_2 - x_1} f(x_1) - \frac{2x_1}{x_2 - x_1} f(x_2). \tag{2}$$

Notice that the trapezoidal rule is a special case of (2). When we choose $x_1 = -1$, $x_2 = 1$, and $h = 2$, then

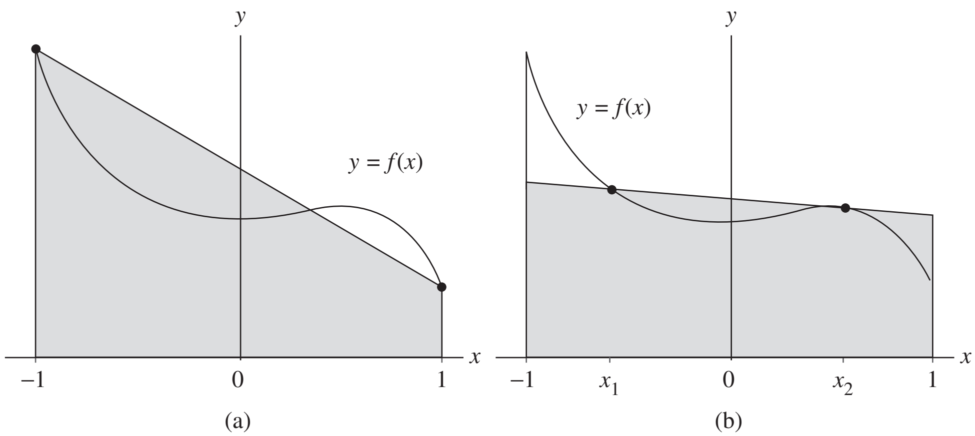

Figure 7.10 (a) Trapezoidal approximation using the abscissas −1 and 1. (b) Trapezoidal approximation using the abscissas x_1 and x_2.

$$T(f, h) = \frac{2}{2} f(x_1) - \frac{-2}{2} f(x_2) = f(x_1) + f(x_2).$$

We shall use the method of undetermined coefficients to find the abscissas x_1, x_2 and weights w_1, w_2 so that the formula

(3) $$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2)$$

is exact for cubic polynomials (i.e., $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$). Since four coefficients w_1, w_2, x_1, and x_2 need to be determined in equation (3), we can select four conditions to be satisfied. Using the fact that integration is additive, it will suffice to require that (3) be exact for the four functions $f(x) = 1, x, x^2, x^3$. The four integral conditions are

(4) $$\begin{aligned} f(x) = 1: &\quad \int_{-1}^{1} 1\,dx = 2 = w_1 + w_2 \\ f(x) = x: &\quad \int_{-1}^{1} x\,dx = 0 = w_1x_1 + w_2x_2 \\ f(x) = x^2: &\quad \int_{-1}^{1} x^2\,dx = \frac{2}{3} = w_1x_1^2 + w_2x_2^2 \\ f(x) = x^3: &\quad \int_{-1}^{1} x^3\,dx = 0 = w_1x_1^3 + w_2x_2^3. \end{aligned}$$

Now solve the system of nonlinear equations

$$w_1 + w_2 = 2 \tag{5}$$

$$w_1 x_1 = -w_2 x_2 \tag{6}$$

$$w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3} \tag{7}$$

$$w_1 x_1^3 = -w_2 x_2^3 \tag{8}$$

We can divide (8) by (6) and the result is

$$x_1^2 = x_2^2 \quad \text{or} \quad x_1 = -x_2. \tag{9}$$

Use (9) and divide (6) by x_1 on the left and $-x_2$ on the right to get

$$w_1 = w_2. \tag{10}$$

Substituting (10) into (5) results in $w_1 + w_2 = 2$. Hence

$$w_1 = w_2 = 1. \tag{11}$$

Now using (11) and (9) in (7), we write

$$w_1 x_1^2 + w_2 x_2^2 = x_2^2 + x_2^2 = \frac{2}{3} \quad \text{or} \quad x_2^2 = \frac{1}{3}. \tag{12}$$

Finally, from (12) and (9) we see that the nodes are

$$-x_1 = x_2 = 1/3^{1/2} \approx 0.5773502692.$$

We have found the nodes and weights that make up the two-point Gauss-Legendre rule. Since the formula is exact for cubic equations, the error term will involve the fourth derivative.

Theorem 7.8 (Gauss-Legendre Two-Point Rule). If f is continuous on $[-1, 1]$, then

$$\int_{-1}^{1} f(x)\,dx \approx G_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right). \tag{13}$$

The Gauss-Legendre rule $G_2(f)$ has degree of precision $n = 3$. If $f \in C^4[-1, 1]$, then

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) + E_2(f), \tag{14}$$

where

$$E_2(f) = \frac{f^{(4)}(c)}{135}. \tag{15}$$

Example 7.17. Use the two-point Gauss-Legendre rule to approximate

$$\int_{-1}^{1} \frac{dx}{x+2} = \ln(3) - \ln(1) \approx 1.09861$$

and compare the result with the trapezoidal rule $T(f, h)$ with $h = 2$ and Simpson's rule $S(f, h)$ with $h = 1$.

Let $G_2(f)$ denote the two-point Gauss-Legendre rule; then

$$\begin{aligned} G_2(f) &= f(-0.57735) + f(0.57735) \\ &= 0.70291 + 0.38800 = 1.09091, \end{aligned}$$

$$\begin{aligned} T(f, 2) &= f(-1.00000) + f(1.00000) \\ &= 1.00000 + 0.33333 = 1.33333, \end{aligned}$$

$$S(f, 1) = \frac{f(-1) + 4f(0) + f(1)}{3} = \frac{1 + 2 + \frac{1}{3}}{3} = 1.11111.$$

The errors are 0.00770, −0.23472, and −0.01250, respectively, so the Gauss-Legendre rule is seen to be best. Notice that the Gauss-Legendre rule required only two function evaluations and Simpson's rule required three. In this example the size of the error for $G_2(f)$ is about 61% of the size of the error for $S(f, 1)$. ■

The general N-point Gauss-Legendre rule is exact for polynomial functions of degree $\leq 2N - 1$, and the numerical integration formula is

$$G_N(f) = w_{N,1} f(x_{N,1}) + w_{N,2} f(x_{N,2}) + \cdots + w_{N,N} f(x_{N,N}). \tag{16}$$

The abscissas $x_{N,k}$ and weights $w_{N,k}$ to be used have been tabulated and are easily available; Table 7.9 gives the values up to eight points. Also included in the table is the form of the error term $E_N(f)$ that corresponds to $G_N(f)$, and it can be used to determine the accuracy of the Gauss-Legendre integration formula.

The values in Table 7.9 in general have no easy representation. This fact makes the method less attractive for humans to use when hand calculations are required. But once the values are stored in a computer it is easy to call them up when needed. The nodes are actually roots of the Legendre polynomials, and the corresponding weights must be obtained by solving a system of equations. For the three-point Gauss-Legendre rule the nodes are $-(0.6)^{1/2}$, 0, and $(0.6)^{1/2}$, and the corresponding weights are 5/9, 8/9, and 5/9.

Table 7.9 Gauss-Legendre Abscissas and Weights

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{N} w_{N,k} f(x_{N,k}) + E_N(f)$$

N	Abscissas, $x_{N,k}$	Weights, $w_{N,k}$	Truncation error, $E_N(f)$
2	-0.5773502692 0.5773502692	1.0000000000 1.0000000000	$\dfrac{f^{(4)}(c)}{135}$
3	± 0.7745966692 0.0000000000	0.5555555556 0.8888888888	$\dfrac{f^{(6)}(c)}{15{,}750}$
4	± 0.8611363116 ± 0.3399810436	0.3478548451 0.6521451549	$\dfrac{f^{(8)}(c)}{3{,}472{,}875}$
5	± 0.9061798459 ± 0.5384693101 0.0000000000	0.2369268851 0.4786286705 0.5688888888	$\dfrac{f^{(10)}(c)}{1{,}237{,}732{,}650}$
6	± 0.9324695142 ± 0.6612093865 ± 0.2386191861	0.1713244924 0.3607615730 0.4679139346	$\dfrac{f^{(12)}(c)2^{13}(6!)^4}{(12!)^3 13!}$
7	± 0.9491079123 ± 0.7415311856 ± 0.4058451514 0.0000000000	0.1294849662 0.2797053915 0.3818300505 0.4179591837	$\dfrac{f^{(14)}(c)2^{15}(7!)^4}{(14!)^3 15!}$
8	± 0.9602898565 ± 0.7966664774 ± 0.5255324099 ± 0.1834346425	0.1012285363 0.2223810345 0.3137066459 0.3626837834	$\dfrac{f^{(16)}(c)2^{17}(8!)^4}{(16!)^3 17!}$

Theorem 7.9 (Gauss-Legendre Three-Point Rule). If f is continuous on $[-1, 1]$, then

$$\int_{-1}^{1} f(x)\,dx \approx G_3(f) = \frac{5f(-\sqrt{3/5}) + 8f(0) + 5f(\sqrt{3/5})}{9}. \tag{17}$$

The Gauss-Legendre rule $G_3(f)$ has degree of precision $n = 5$. If $f \in C^6[-1, 1]$, then

$$\int_{-1}^{1} f(x)\,dx = \frac{5f(-\sqrt{3/5}) + 8f(0) + 5f(\sqrt{3/5})}{9} + E_3(f), \tag{18}$$

where

$$E_3(f) = \frac{f^{(6)}(c)}{15,750}. \tag{19}$$

Example 7.18. Show that the three-point Gauss-Legendre rule is exact for

$$\int_{-1}^{1} 5x^4\,dx = 2 = G_3(f).$$

Since the integrand is $f(x) = 5x^4$ and $f^{(6)}(x) = 0$, we can use (19) to see that $E_3(f) = 0$. But it is instructive to use (17) and do the calculations in this case.

$$G_3(f) = \frac{5(5)(0.6)^2 + 0 + 5(5)(0.6)^2}{9} = \frac{18}{9} = 2. \qquad \blacksquare$$

The next result shows how to change the variable of integration so that the Gauss-Legendre rules can be used on the interval $[a, b]$.

Theorem 7.10 (Gauss-Legendre Translation). Suppose that the abscissas $\{x_{N,k}\}_{k=1}^{N}$ and weights $\{w_{N,k}\}_{k=1}^{N}$ are given for the N-point Gauss-Legendre rule over $[-1, 1]$. To apply the rule over the interval $[a, b]$, use the change of variable

$$\text{(20)} \qquad t = \frac{a+b}{2} + \frac{b-a}{2}x \quad \text{and} \quad dt = \frac{b-a}{2}\,dx.$$

Then the relationship

$$\text{(21)} \qquad \int_a^b f(t)\,dt = \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}x\right)\frac{b-a}{2}\,dx$$

is used to obtain the quadrature formula

$$\text{(22)} \qquad \int_a^b f(t)\,dt = \frac{b-a}{2}\sum_{k=1}^{N} w_{N,k}\, f\left(\frac{a+b}{2} + \frac{b-a}{2}x_{N,k}\right).$$

Example 7.19. Use the three-point Gauss-Legendre rule to approximate

$$\int_1^5 \frac{dt}{t} = \ln(5) - \ln(1) \approx 1.609438$$

and compare the result with Boole's rule $B(2)$ with $h = 1$.

Here $a = 1$ and $b = 5$, so the rule in (22) yields

$$\begin{aligned} G_3(f) &= (2)\frac{5f(3 - 2(0.6)^{1/2}) + 8f(3+0) + 5f(3 + 2(0.6)^{1/2})}{9} \\ &= (2)\frac{3.446359 + 2.666667 + 1.099096}{9} = 1.602694. \end{aligned}$$

In Example 7.13 we saw that Boole's rule gave $B(2) = 1.617778$. The errors are 0.006744 and −0.008340, respectively, so that the Gauss-Legendre rule is slightly better in this case. Notice that the Gauss-Legendre rule requires three function evaluations and Boole's rule requires five. In this example the size of the two errors is about the same. ■

Gauss-Legendre integration formulas are extremely accurate, and they should be considered seriously when many integrals of a similar nature are to be evaluated. In this case, proceed as follows. Pick a few representative integrals, including some with the worst behavior that is likely to occur. Determine the number of sample points N that is needed to obtain the required accuracy. Then fix the value N, and use the Gauss-Legendre rule with N sample points for all the integrals.

Numerical Methods Using Matlab, 4th Edition, 2004

John H. Mathews and Kurtis K. Fink

ISBN: 0-13-065248-2

Prentice-Hall Inc.
Upper Saddle River, New Jersey, USA
http://vig.prenhall.com/