We close this section with a result that will help us gain information about the location of the zeros and poles of meromorphic functions.

**Theorem 8.10 (Rouché’s Theorem)** Suppose that \( f \) and \( g \) are meromorphic functions defined in the simply connected domain \( D \), that \( C \) is a simply closed contour in \( D \), and that \( f \) and \( g \) have no zeros or poles for \( z \in C \). If the strict inequality \(|f(z) + g(z)| < |f(z)| + |g(z)|\) holds for all \( z \in C \), then \( Z_f - P_f = Z_g - P_g \).

**Proof** Because \( g \) has no zeros or poles on \( C \), we may legitimately divide both sides of the inequality \(|f(z) + g(z)| < |f(z)| + |g(z)|\) by \(|g(z)|\) to get

\[
\left| \frac{f(z)}{g(z)} + 1 \right| < \left| \frac{f(z)}{g(z)} \right| + 1, \quad \text{for all } z \in C. \tag{8-37}
\]

For \( z \in C \), \( \frac{f(z)}{g(z)} \) cannot possibly be zero or any positive real number, as that would contradict Inequality (8-37). This means that \( C^* \), the image of the curve \( C \) under the mapping \( \frac{f}{g} \), does not contain the interval \([0, \infty)\), and so the function defined by

\[
w(z) = \log_0 \left( \frac{f(z)}{g(z)} \right) = \ln \left| \frac{f(z)}{g(z)} \right| + i \arg_0 \left( \frac{f(z)}{g(z)} \right) = \ln r + i \phi,
\]
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where \( f(z) = re^{i\phi} \neq 0 \) and \( 0 < \phi \leq 2\pi \), is analytic in a simply connected domain \( D^* \) that contains \( C^* \). We calculate

\[ w'(z) = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}, \]

so \( w(z) = \log \left( \frac{f(z)g(z)}{g(z)} \right) \) is an antiderivative of \( f'(z)/f(z) - g'(z)/g(z) \), for all \( z \in D^* \). As \( C^* \) is a closed curve in \( D^* \), Theorem 6.9 gives

\[ \int_{C^*} \left( \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} \right) \, dz = 0. \]

According to Theorem 8.8, then

\[ \int_{C^*} \frac{f'(z)}{f(z)} \, dz - \int_{C^*} \frac{g'(z)}{g(z)} \, dz = (Z_f - P_f) - (Z_g - P_g) = 0, \]

which completes the proof.

\textbf{Corollary 8.2} Suppose that \( f \) and \( g \) are analytic functions defined in the simply connected domain \( D \), that \( C \) is a simple closed contour in \( D \), and that \( f \) and \( g \) have no zeros for \( z \in C \). If the strict inequality \( |f(z) + g(z)| < |f(z)| + |g(z)| \) holds for all \( z \in C \), then \( Z_f = Z_g \).

\textbf{Remark 8.5} Theorem 8.10 is usually stated with the requirement that \( f \) and \( g \) satisfy the condition \( |f(z) + g(z)| < |g(z)| \), for \( z \in C \). The improved theorem that we gave was discovered by Irving Glicksberg (see the American Mathematical Monthly, 83 (1976), pp. 186–187). The weaker version is adequate for most purposes, however, as the following examples illustrate.