If \( f \) has a removable singularity at \( z_0 \), then \( a_{-n} = 0 \), for \( n = 1, 2, \ldots \). Therefore, \( \text{Res}[f, z_0] = 0 \). Theorem 8.2 gives methods for evaluating residues at poles.

**Theorem 8.2 (Residues at poles)**

i. If \( f \) has a simple pole at \( z_0 \), then
\[
\text{Res}[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z).
\]

ii. If \( f \) has a pole of order 2 at \( z_0 \), then
\[
\text{Res}[f, z_0] = \lim_{z \to z_0} \frac{d}{dz} (z - z_0)^2 f(z).
\]

iii. If \( f \) has a pole of order \( k \) at \( z_0 \), then
\[
\text{Res}[f, z_0] = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} (z - z_0)^k f(z).
\]

**Proof**

If \( f \) has a simple pole at \( z_0 \), then the Laurent series is
\[
f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots.
\]

If we multiply both sides of this equation by \( (z - z_0) \) and take the limit as \( z \to z_0 \), we obtain
\[
\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \left[ a_{-1} + a_0 (z - z_0) + a_1 (z - z_0)^2 + \cdots \right] = a_{-1} = \text{Res}[f, z_0],
\]
which establishes part (i). We proceed to part (iii), as part (ii) is a special case of it. Suppose that \(f\) has a pole of order \(k\) at \(z_0\). Then \(f\) can be written as

\[
f(z) = \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots.
\]

Multiplying both sides of this equation by \((z-z_0)^k\) gives

\[
(z-z_0)^k f(z) = a_{-k} + \cdots + a_{-1}(z-z_0)^{k-1} + a_0(z-z_0)^k + \cdots.
\]

If we differentiate both sides \(k-1\) times, we get

\[
\frac{d^{k-1}}{dz^{k-1}} \left[(z-z_0)^k f(z)\right] = (k-1)!a_{-1} + k!a_0(z-z_0)
\]

\[
+ (k+1)!a_1(z-z_0)^2 + \cdots,
\]

and when we let \(z \to z_0\), the result is

\[
\lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_0)^k f(z)\right] = (k-1)!a_{-1} = (k-1)! \text{Res}[f, z_0],
\]

which establishes part (iii).