Theorem 8.1 (Cauchy's residue theorem) *Let D be a simply connected domain and let C be a simple closed positively oriented contour that lies in D. If f is analytic inside C and on C, except at the points z_1, z_2,..., z_n that lie inside C, then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res[f, z_k].$$

The situation is illustrated in Figure 8.1.

Proof Since there are a finite number of singular points inside C, there exists an $r > 0$ such that the positively oriented circles $C_k = C_r^+(z_k)$, for $k = 1, 2, \ldots, n$, are mutually disjoint and all lie inside C. From the extended Cauchy–Goursat theorem (Theorem 6.7), it follows that

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz.$$

The function f is analytic in a punctured disk with center z_k that contains the circle C_k, so we can use Equation (8-2) to obtain

$$\int_{C_k} f(z)\,dz = 2\pi i \operatorname{Res}[f, z_k], \qquad \text{for } k = 1, 2, \ldots, n.$$

Combining the last two equations gives the desired result.

The calculation of a Laurent series expansion is tedious in most circumstances. Since the residue at z_0 involves only the coefficient a_{-1} in the Laurent

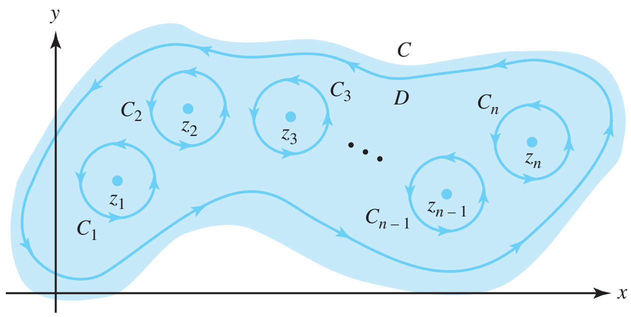

Figure 8.1 The domain D and contour C and the singular points z_1, z_2,..., z_n in the statement of Cauchy's residue theorem.

expansion, we seek a method to calculate the residue from special information about the nature of the singularity at z_0.