We now state a general result that shows how to accomplish differentiation under the integral sign. The proof is presented in some advanced texts. See, for instance, Rolf Nevanlinna and V. Paatero, *Introduction to Complex Analysis* (Reading, Mass.: Addison-Wesley, 1969), Section 9.7.

**Theorem 6.11 (Leibniz’s rule)** Let $G$ be an open set and let $I : a \leq t \leq b$ be an interval of real numbers. Let $g(z,t)$ and its partial derivative $g_z(z,t)$ with respect to $z$ be continuous functions for all $z$ in $G$ and all $t$ in $I$. Then $F(z) = \int_a^b g(z,t) \, dt$ is analytic for $z$ in $G$, and $F'(z) = \int_a^b g_z(z,t) \, dt$.

We now generalize Theorem 6.10 to give an integral representation for the $n$th derivative, $f^{(n)}(z)$. We use Leibniz’s rule in the proof and note that this method of proof is a mnemonic device for remembering Theorem 6.12.

**Theorem 6.12 (Cauchy’s integral formulas for derivatives)** Let $f$ be analytic in the simply connected domain $D$ and let $C$ be a simple closed positively oriented contour that lies in $D$. If $z$ is a point that lies interior to $C$, then for any integer $n \geq 0$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi.$$  \hfill (6-48)

**Proof** Because $f^{(0)}(z) = f(z)$, the case for $n = 0$ reduces to Theorem 6.10. We now establish the theorem for the case $n = 1$. We start by using the parametrization

$C : \xi = \xi(t)$ and $d\xi = \xi'(t) \, dt$, for $a \leq t \leq b$.

We use Theorem 6.10 and write

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} \, d\xi = \frac{1}{2\pi i} \int_a^b \frac{f(\xi(t))\xi'(t)}{\xi(t) - z} \, dt.$$  \hfill (6-49)
The integrand on the right side of Equation (6-49) is a function \( g(z, t) \) of the two variables \( z \) and \( t \), where

\[
g(z, t) = \frac{f(\xi(t)) \xi'(t)}{\xi(t) - z} \quad \text{and} \quad \frac{\partial g}{\partial z}(z, t) = g_z(z, t) = \frac{f(\xi(t)) \xi'(t)}{(\xi(t) - z)^2}.
\]

Moreover, \( g(z, t) \) and \( g_z(z, t) \) are continuous on the interior of \( C \), which is an open set. Applying Leibniz’s rule to Equations (6-49) gives

\[
f'(z) = \frac{1}{2\pi i} \int_a^b \frac{f(\xi(t)) \xi'(t) \, dt}{(\xi(t) - z)^2} = \frac{1}{2\pi i} \int_C \frac{f(\xi) \, d\xi}{(\xi - z)^2},
\]

and the proof for the case \( n = 1 \) is complete. We can apply the same argument to the analytic function \( f' \) and show that its derivative \( f'' \) is also represented by Equation (6-48) for \( n = 2 \). The principle of mathematical induction establishes the theorem for all integers \( n \geq 0 \).