We now state a general result that shows how to accomplish differentiation under the integral sign. The proof is presented in some advanced texts. See, for instance, Rolf Nevanlinna and V. Paatero, *Introduction to Complex Analysis* (Reading, Mass.: Addison-Wesley, 1969), Section 9.7.

Theorem 6.11 (Leibniz's rule) *Let G be an open set and let $I : a \le t \le b$ be an interval of real numbers. Let $g(z,t)$ and its partial derivative $g_z(z,t)$ with respect to z be continuous functions for all z in G and all t in I. Then $F(z) = \int_a^b g(z,t)\,dt$ is analytic for z in G, and $F'(z) = \int_a^b g_z(z,t)\,dt$.*

We now generalize Theorem 6.10 to give an integral representation for the nth derivative, $f^{(n)}(z)$. We use Leibniz's rule in the proof and note that this method of proof is a mnemonic device for remembering Theorem 6.12.

Theorem 6.12 (Cauchy's integral formulas for derivatives) *Let f be analytic in the simply connected domain D and let C be a simple closed positively oriented contour that lies in D. If z is a point that lies interior to C, then for any integer $n \ge 0$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi. \tag{6-48}$$

Proof Because $f^{(0)}(z) = f(z)$, the case for $n = 0$ reduces to Theorem 6.10. We now establish the theorem for the case $n = 1$. We start by using the parametrization

$$C : \xi = \xi(t) \quad \text{and} \quad d\xi = \xi'(t)\,dt, \qquad \text{for } a \le t \le b.$$

We use Theorem 6.10 and write

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i} \int_a^b \frac{f(\xi(t))\,\xi'(t)\,dt}{\xi(t) - z}. \tag{6-49}$$

The integrand on the right side of Equation (6-49) is a function $g(z,t)$ of the two variables z and t, where

$$g(z,\, t) = \frac{f(\xi(t))\, \xi'(t)}{\xi(t) - z} \quad \text{and} \quad \frac{\partial g}{dz}(z,t) = g_z(z,\, t) = \frac{f(\xi(t))\, \xi'(t)}{(\xi(t) - z)^2}.$$

Moreover, $g(z,t)$ and $g_z(z,t)$ are continuous on the interior of C, which is an open set. Applying Leibniz's rule to Equations (6-49) gives

$$f'(z) = \frac{1}{2\pi i} \int_a^b \frac{f(\xi(t))\, \xi'(t)\, dt}{(\xi(t) - z)^2} = \frac{1}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi - z)^2},$$

and the proof for the case $n = 1$ is complete. We can apply the same argument to the analytic function f' and show that its derivative f'' is also represented by Equation (6-48) for $n = 2$. The principle of mathematical induction establishes the theorem for all integers $n \geq 0$.