6.5 INTEGRAL REPRESENTATIONS FOR ANALYTIC FUNCTIONS

We now present some major results in the theory of functions of a complex variable. The first result is known as Cauchy's integral formula and shows that the value of an analytic function f can be represented by a certain contour integral. The nth derivative, $f^{(n)}(z)$, will have a similar representation. In Chapter 7, we use the Cauchy integral formulas to prove Taylor's theorem and also establish the power series representation for analytic functions. The Cauchy integral formulas are a convenient tool for evaluating certain contour integrals.

Theorem 6.10 (Cauchy's integral formula) *Let f be analytic in the simply connected domain D and let C be a simple closed positively oriented contour that lies in D. If z_0 is a point that lies interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz. \tag{6-44}$$

Proof Because f is continuous at z_0, if $\varepsilon > 0$ is given, there is a $\delta > 0$ such that the positively oriented circle $C_0 = \left\{z : |z - z_0| = \frac{1}{2}\delta\right\}$ lies interior to C (as Figure 6.33 shows) and such that

$$|f(z) - f(z_0)| < \varepsilon, \qquad \text{whenever } |z - z_0| < \delta. \tag{6-45}$$

Since $f(z_0)$ is a fixed value, we can use the result of Corollary 6.1 to conclude that

$$f(z_0) = \frac{f(z_0)}{2\pi i}\int_{C_0} \frac{dz}{z - z_0} = \frac{1}{2\pi i}\int_{C_0} \frac{f(z_0)}{z - z_0}\,dz. \tag{6-46}$$

By the deformation of contour theorem (Theorem 6.6),

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi i}\int_{C_0} \frac{f(z)}{z - z_0}\,dz. \tag{6-47}$$

Using Inequality (6-45) and Equations (6-46) and (6-47) above, together with the ML inequality (Theorem 6.3), we obtain the estimate:

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0} - f(z_0)\right| &= \left|\frac{1}{2\pi i}\int_{C_0} \frac{f(z)\,dz}{z - z_0} - \frac{1}{2\pi i}\int_{C_0} \frac{f(z_0)\,dz}{z - z_0}\right| \\ &\le \frac{1}{2\pi}\int_{C_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}\,|dz| \\ &\le \frac{1}{2\pi}\frac{\varepsilon}{\left(\frac{1}{2}\right)\delta}\pi\delta = \varepsilon.\end{aligned}$$

This proves the theorem because ε can be made arbitrarily small.