We now present some major results in the theory of functions of a complex variable. The first result is known as Cauchy’s integral formula and shows that the value of an analytic function \( f \) can be represented by a certain contour integral. The \( n \)th derivative, \( f^{(n)}(z) \), will have a similar representation. In Chapter 7, we use the Cauchy integral formulas to prove Taylor’s theorem and also establish the power series representation for analytic functions. The Cauchy integral formulas are a convenient tool for evaluating certain contour integrals.

\[ f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} \, dz. \]  

\((6-44)\)

**Proof**  
Because \( f \) is continuous at \( z_0 \), if \( \varepsilon > 0 \) is given, there is a \( \delta > 0 \) such that the positively oriented circle \( C_0 = \{ z : |z - z_0| = \frac{1}{2} \delta \} \) lies interior to \( C \) (as Figure 6.33 shows) and such that  

\[ |f(z) - f(z_0)| < \varepsilon, \quad \text{whenever } |z - z_0| < \delta. \]  

\((6-45)\)

Since \( f(z_0) \) is a fixed value, we can use the result of Corollary 6.1 to conclude that  

\[ f(z_0) = \frac{f(z_0)}{2\pi i} \int_{C_0} \frac{dz}{z-z_0} = \frac{1}{2\pi i} \int_{C_0} \frac{f(z_0)}{z-z_0} \, dz. \]  

\((6-46)\)

By the deformation of contour theorem (Theorem 6.6),  

\[ \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} \, dz = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{z-z_0} \, dz. \]  

\((6-47)\)

Using Inequality \((6-45)\) and Equations \((6-46)\) and \((6-47)\) above, together with the \( ML \) inequality (Theorem 6.3), we obtain the estimate:  

\[ \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} \, dz - f(z_0) \right| = \left| \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{z-z_0} \, dz - \frac{1}{2\pi i} \int_{C_0} \frac{f(z_0)}{z-z_0} \, dz \right| \leq \frac{1}{2\pi} \int_{C_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} |dz| \leq \frac{1}{2\pi} \frac{\varepsilon}{\left( \frac{1}{2} \right) \delta} \pi \delta = \varepsilon. \]

This proves the theorem because \( \varepsilon \) can be made arbitrarily small.