We now state two important corollaries of Theorem 6.12.

◗ **Corollary 6.2** If f is analytic in the domain D, then, for integers $n \geq 0$, all derivatives $f^{(n)}(z)$ exist for $z \in D$ (and therefore are analytic in D).

Proof For each point z_0 in D, there exists a closed disk $|z - z_0| \leq R$ that is contained in D. We use the circle $C = C_R(z_0) = \{z : |z - z_0| = R\}$ in Theorem 6.12 to show that $f^{(n)}(z_0)$ exists for all integers $n \geq 0$.

Remark 6.3 This result is interesting, as it illustrates a big difference between real and complex functions. A real function f can have the property that f' exists everywhere in a domain D, but f'' exists nowhere. Corollary 6.2 states that if a complex function f has the property that f' exists everywhere in a domain D, then, remarkably, *all* derivatives of f exist in D. ■

◗ **Corollary 6.3** If u is a harmonic function at each point (x, y) in the domain D, then all partial derivatives u_x, u_y, u_{xx}, u_{xy}, and u_{yy} exist and are harmonic functions.

Proof For each point $z_0 = (x_0,\ y_0)$ in D there exists a disk $D_R(z_0)$ that is contained in D. In this disk, a conjugate harmonic function v exists, so the function $f(z) = u + iv$ is analytic. We use the Cauchy–Riemann equations to get $f'(z) = u_x + iv_x = v_y - iu_y$, for $z \in D_R(z_0)$. Since f' is analytic in $D_R(z_0)$, the functions u_x and u_y are harmonic there. Again, we can use the Cauchy–Riemann equations to obtain, for $z \in D_R(z_0)$,

$$f''(z) = u_{xx} + iv_{xx} = v_{yx} - iu_{yx} = -u_{yy} - iv_{yy}.$$

Because f'' is analytic in $D_R(z_0)$, the functions u_{xx}, u_{xy}, and u_{yy} are harmonic there.