We now state two important corollaries of Theorem 6.12.

**Corollary 6.2** If \( f \) is analytic in the domain \( D \), then, for integers \( n \geq 0 \), all derivatives \( f^{(n)}(z) \) exist for \( z \in D \) (and therefore are analytic in \( D \)).

**Proof** For each point \( z_0 \) in \( D \), there exists a closed disk \( |z - z_0| \leq R \) that is contained in \( D \). We use the circle \( C = C_R(z_0) = \{ z : |z - z_0| = R \} \) in Theorem 6.12 to show that \( f^{(n)}(z_0) \) exists for all integers \( n \geq 0 \).

**Remark 6.3** This result is interesting, as it illustrates a big difference between real and complex functions. A real function \( f \) can have the property that \( f' \) exists everywhere in a domain \( D \), but \( f'' \) exists nowhere. Corollary 6.2 states that if a complex function \( f \) has the property that \( f' \) exists everywhere in a domain \( D \), then, remarkably, all derivatives of \( f \) exist in \( D \).

**Corollary 6.3** If \( u \) is a harmonic function at each point \( (x, y) \) in the domain \( D \), then all partial derivatives \( u_x, u_y, u_{xx}, u_{xy}, \) and \( u_{yy} \) exist and are harmonic functions.

**Proof** For each point \( z_0 = (x_0, y_0) \) in \( D \) there exists a disk \( D_R(z_0) \) that is contained in \( D \). In this disk, a conjugate harmonic function \( v \) exists, so the function \( f(z) = u + iv \) is analytic. We use the Cauchy–Riemann equations to get \( f'(z) = u_x + iv_x = v_y - iu_y \), for \( z \in D_R(z_0) \). Since \( f' \) is analytic in \( D_R(z_0) \), the functions \( u_x \) and \( u_y \) are harmonic there. Again, we can use the Cauchy–Riemann equations to obtain, for \( z \in D_R(z_0) \),

\[
f''(z) = u_{xx} + iv_{xx} = v_{yy} - iu_{yy} = -u_{yy} - iv_{yy}.
\]

Because \( f'' \) is analytic in \( D_R(z_0) \), the functions \( u_{xx}, u_{xy}, \) and \( u_{yy} \) are harmonic there.