6.4 THE FUNDAMENTAL THEOREMS OF INTEGRATION

Let $f$ be analytic in the simply connected domain $D$. The theorems in this section show that an antiderivative $F$ can be constructed by contour integration. A consequence will be the fact that in a simply connected domain, the integral of an analytic function $f$ along any contour joining $z_1$ to $z_2$ is the same, and its value is given by $F(z_2) - F(z_1)$. As a result, we can use the antiderivative formulas from calculus to compute the value of definite integrals.

**Theorem 6.8 (Indefinite integrals, or antiderivatives)** Let $f$ be analytic in the simply connected domain $D$. If $z_0$ is a fixed value in $D$ and if $C$ is any contour in $D$ with initial point $z_0$ and terminal point $z$, then the function

$$F(z) = \int_C f(\xi) \, d\xi = \int_{z_0}^{z} f(\xi) \, d\xi$$

(6-40)

is well-defined and analytic in $D$, with its derivative given by $F'(z) = f(z)$.

**Proof** We first establish that the integral is independent of the path of integration. This will show that the function $F$ is well-defined, which in turn will justify the notation $F(z) = \int_{z_0}^{z} f(\xi) \, d\xi$. 
We let \( C_1 \) and \( C_2 \) be two contours in \( D \), both with initial point \( z_0 \) and terminal point \( z \), as shown in Figure 6.30. Then \( C_1 - C_2 \) is a simple closed contour, and the Cauchy–Goursat theorem implies that

\[
\int_{C_1} f(\xi) \, d\xi - \int_{C_2} f(\xi) \, d\xi = \int_{C_1 - C_2} f(\xi) \, d\xi = 0.
\]

Therefore, the contour integral in Equation (6-40) is independent of path. Here we have taken the liberty of drawing contours that intersect only at the endpoints. A slight modification of the proof shows that a finite number of other points of intersection are permitted.

We now show that \( F'(z) = f(z) \). Let \( z \) be held fixed, and let \( |\Delta z| \) be chosen small enough so that the point \( z + \Delta z \) also lies in the domain \( D \). Since \( z \) is held fixed, \( f(z) = K \), where \( K \) is a constant, and Equation (6-9) implies that

\[
\int_{z}^{z+\Delta z} f (\xi) \, d\xi = \int_{z}^{z+\Delta z} K \, d\xi = K \Delta z = f(\xi) \Delta z . \tag{6-41}
\]

Using the additive property of contours and the definition of \( F \) given in Equation (6-40), we have

\[
F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(\xi) \, d\xi - \int_{z_0}^{z} f(\xi) \, d\xi
= \int_{\Gamma_2} f(\xi) \, d\xi - \int_{\Gamma_1} f(\xi) \, d\xi = \int_{\Gamma} f(\xi) \, d\xi , \tag{6-42}
\]

where the contour \( \Gamma \) is the straight-line segment joining \( z \) to \( z + \Delta z \), and \( \Gamma_1 \) and \( \Gamma_2 \) join \( z_0 \) to \( z \), and \( z_0 \) to \( z + \Delta z \), respectively, as shown in Figure 6.31. Since \( f \) is continuous at \( z \), for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that

\[
| f(\xi) - f(z) | < \varepsilon \quad \text{when } |\xi - z| < \delta.
\]
If we require that $|\Delta z| < \delta$ and combine this last inequality with Equations (6.41) and (6.42), and Theorem (6.23), we get

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|} \left| \int_{\Gamma} f(\xi) \, d\xi - \int_{\Gamma} f(z) \, d\xi \right|$$

$$\leq \frac{1}{|\Delta z|} \int_{\Gamma} |f(\xi) - f(z)| \, |d\xi|$$

$$< \frac{1}{|\Delta z|^2} |\Delta z| = \varepsilon.$$ 

Thus, $\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right|$ tends to 0 as $\Delta z \to 0$, so $F'(z) = f(z)$.

**Remark 6.2** It is important to stress that the line integral of an analytic function is independent of path. In Example 6.9 we showed that $\int_{C_1} z \, dz = \int_{C_2} z \, dz = 4 + 2i$, where $C_1$ and $C_2$ were different contours joining $-1 - i$ to $3 + i$. Because the integrand $f(z) = z$ is an analytic function, Theorem 6.8 lets us know ahead of time that the value of the two integrals is the same; hence one calculation would have sufficed. If you ever have to compute a line integral of an analytic function over a difficult contour, change the contour to something easier. You are guaranteed to get the same answer. Of course, you must be sure that the function you’re dealing with is analytic in a simply connected domain containing your original and new contours.