12.2 THE DIRICHLET PROBLEM FOR THE UNIT DISK

The Dirichlet problem for the unit disk \( D : |z| < 1 \) is to find a real-valued function \( u(x, y) \) that is harmonic in the unit disk \( D \) and that takes on the boundary values

\[
u(\cos \theta, \sin \theta) = U(\theta), \quad \text{for } -\pi < \theta \leq \pi, \tag{12-10}\]

at points \( z = (\cos \theta, \sin \theta) \) on the unit circle, as shown in Figure 12.12.

\[\blacktriangleright\textbf{Theorem 12.7} \quad \text{If } U(t) \text{ has period } 2\pi \text{ and has the Fourier series representation}
\]

\[
U(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left( a_j \cos jt + b_j \sin jt \right),
\]

\[\text{then the solution } u \text{ to the Dirichlet problem in } D \text{ is}
\]

\[
u(r \cos \theta, r \sin \theta) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left( a_j r^j \cos j\theta + b_j r^j \sin j\theta \right), \tag{12-11}\]

\[\text{where } z = x + iy = re^{i\theta} \text{ denotes a complex number in the closed disk } |z| \leq 1.\]

The series representation in Equation (12-11) for \( u \) takes on the prescribed boundary values in Equation (12-10) at points on the unit circle \( |z| = 1 \). Each
Theorem 12.8 (Poisson integral formula for the unit disk) Let \( u(x, y) \) be a function that is harmonic in a simply connected domain that contains the closed unit disk \(|z| \leq 1\). If \( u(x, y) \) takes on the boundary values

\[
 u(\cos \theta, \sin \theta) = U(\theta), \quad \text{for } -\pi < \theta \leq \pi,
\]

then \( u \) has the integral representation

\[
 u(r \cos \theta, r \sin \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) U(t) dt}{1 + r^2 - 2r \cos(t - \theta)},
\]  \hspace{1cm} (12-12)

which is valid for \(|z| < 1\).

**Proof** Since \( u(x, y) \) is harmonic in the simply connected domain, there exists a conjugate harmonic function \( v(x, y) \) such that \( f(z) = u(x, y) + iv(x, y) \) is
analytic. Let \( C \) denote the contour consisting of the unit circle; then Cauchy’s integral formula

\[
f (z) = \frac{1}{2\pi i} \int_C \frac{f (\xi)}{\xi - z} \quad (12-13)
\]

expresses the value of \( f (z) \) at any point \( z \) inside \( C \) in terms of the values of \( f (\xi) \) at points \( \xi \) that lie on the circle \( C \).

If we set \( z^* = (\pi)^{-1} \), then \( z^* \) lies outside the unit circle \( C \) and the Cauchy–Goursat theorem establishes the equation

\[
0 = \frac{1}{2\pi i} \int_C \frac{f (\xi)}{\xi - z^*} \quad (12-14)
\]

Subtracting Equation (12-14) from Equation (12-13) and using the parameterization \( \xi = e^{it} \), \( d\xi = ie^{it} dt \) and the substitutions \( z = re^{i\theta} \), \( z^* = \frac{1}{r}e^{i\theta} \) gives

\[
f (z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{e^{it}}{e^{it} - re^{i\theta}} - \frac{e^{it}}{e^{it} - \frac{1}{r}e^{i\theta}} \right) f (e^{it}) \, dt.
\]

We rewrite the expression inside the parentheses on the right side of this equation as

\[
\frac{e^{it}}{e^{it} - re^{i\theta}} - \frac{e^{it}}{e^{it} - \frac{1}{r}e^{i\theta}} = \frac{1}{1 - re^{i(\theta-t)}} + \frac{re^{i(t-\theta)}}{1 - re^{i(\theta-t)}}
\]

\[
= \frac{1 - r^2}{1 + r^2 - 2r \cos(t - \theta)},
\]

and it follows that

\[
f (z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(t - \theta)} f (e^{it}) \, dt.
\]

Because \( u(x,y) \) is the real part of \( f (z) \) and \( U(t) \) is the real part of \( f (e^{it}) \), we can equate the real parts in the preceding equation to obtain Equation (12-12), completing the proof of Theorem 12.8.

We now turn to the proof Theorem 12.7. The real-valued function

\[
P (r, t - \theta) = \frac{1 - r^2}{1 + r^2 - 2r \cos(t - \theta)}
\]
is known as the **Poisson kernel**. Expanding the left side of Equation (12-15) in a geometric series gives

\[
P(r, t - \theta) = \frac{1}{1 - re^{i(\theta - t)}} + \frac{re^{i(t - \theta)}}{1 - re^{i(\theta - t)}} = \sum_{n=0}^{\infty} r^n e^{in(\theta - t)} + \sum_{n=1}^{\infty} r^n e^{in(t - \theta)}
\]

\[
= 1 + \sum_{n=1}^{\infty} r^n \left[ e^{in(\theta - t)} + e^{in(t - \theta)} \right] = 1 + 2 \sum_{n=1}^{\infty} r^n \cos[n(\theta - t)]
\]

\[
= 1 + 2 \sum_{n=1}^{\infty} r^n \cos n\theta \cos nt + 2 \sum_{n=1}^{\infty} r^n \sin n\theta \sin nt.
\]

We now use this result in Equation (12-12) to obtain

\[
u(r \cos \theta, r \sin \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, t - \theta) U(t) \, dt
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} U(t) \, dt + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} r^n \cos n\theta \cos nt \, U(t) \, dt
\]

\[
+ \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} r^n \sin n\theta \cos nt \, U(t) \, dt
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} U(t) \, dt + \sum_{n=1}^{\infty} \frac{r^n}{\pi} \cos n\theta \int_{-\pi}^{\pi} \cos nt \, U(t) \, dt
\]

\[
+ \sum_{n=1}^{\infty} \frac{r^n}{\pi} \sin n\theta \int_{-\pi}^{\pi} \sin nt \, U(t) \, dt
\]

\[
= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + \sum_{n=1}^{\infty} b_n r^n \sin n\theta,
\]

where \(\{a_n\}\) and \(\{b_n\}\) are the Fourier series coefficients for \(U(t)\). This result establishes the representation for \(u(r \cos \theta, r \sin \theta)\) in Equation (12-11) of Theorem 12.7.