Theorem 6.4 (Green’s theorem) Let $C$ be a simple closed contour with positive orientation and let $R$ be the domain that forms the interior of $C$. If $P$ and $Q$ are continuous and have continuous partial derivatives $P_x$, $P_y$, $Q_x$, and $Q_y$ at all points on $C$ and $R$, then

$$\int_C P(x,y) \, dx + Q(x,y) \, dy = \iint_R \left[ Q_x(x,y) - P_y(x,y) \right] \, dx \, dy. \quad (6-25)$$

Proof (For a standard region.*) If $R$ is a standard region, then there exist functions $y = g_1(x)$, and $y = g_2(x)$, for $a \leq x \leq b$, whose graphs form the lower and upper portions of $C$, respectively, as indicated in Figure 6.18. As $C$ is positively oriented, these functions can be used to express $C$ as the sum of two contours $C_1$ and $C_2$, where

$C_1 : z_1(t) = t + ig_1(t), \quad$ for $a \leq t \leq b$, and

$C_2 : z_2(t) = -t + ig_2(-t), \quad$ for $-b \leq t \leq -a$.

We now use the functions $g_1(x)$ and $g_2(x)$ to express the double integral of $-P_y(x,y)$ over $R$ as an iterated integral, first with respect to $y$ and second with respect to $x$:

$$-\iint_R P_y(x,y) \, dx \, dy = -\int_a^b \left[ \int_{g_1(x)}^{g_2(x)} P_y(x,y) \, dy \right] \, dx.$$

Computing the first iterated integral on the right side gives

$$-\iint_R P_y(x,y) \, dx \, dy = \int_a^b P(x,g_1(x)) \, dx - \int_a^b P(x,g_2(x)) \, dx.$$

In the second integral on the right side of this equation we can use the change of variable $x = -t$ to obtain

$$-\iint_R P_y(x,y) \, dx \, dy = \int_a^b P(x,g_1(x)) \, dx + \int_{-b}^{-a} P(-t,g_2(-t)) \, dt.$$

Interpreting the two integrals on the right side of this equation as contour integrals along $C_1$ and $C_2$, respectively, gives

$$-\iint_R P_y(x,y) \, dx \, dy = \int_{C_1} P(x,y) \, dx + \int_{C_2} P(x,y) \, dx = \int_C P(x,y) \, dx.$$

(6-26)

* A standard region is bounded by a contour $C$, which can be expressed in the two forms $C = C_1 + C_2$ and $C = C_3 + C_4$ that are used in the proof.
To complete the proof, we rely on the fact that for a standard region there exist functions \( x = h_1 (y) \) and \( x = h_2 (y) \) for \( c \leq y \leq d \) whose graphs form the left and right portions of \( C \), respectively, as indicated in Figure 6.19. Because \( C \) has the positive orientation, it can be expressed as the sum of two contours \( C_3 \) and \( C_4 \), where

\[
C_3 : z_3 (t) = h_1 (-t) - it, \quad \text{for} \quad -d \leq t \leq -c, \quad \text{and} \\
C_4 : z_4 (t) = h_2 (t) + it, \quad \text{for} \quad c \leq t \leq d.
\]

Using the functions \( h_1 (y) \) and \( h_2 (y) \), we express the double integral of \( Q_x (x,y) \) over \( R \) as an iterated integral:

\[
\int\int_R Q_x (x,y) \, dx \, dy = \int_c^d \left[ \int_{h_1 (y)}^{h_2 (y)} Q_x (x,y) \, dx \right] \, dy.
\]

A derivation similar to that which led to Equation (6-26) shows that

\[
\int\int_R Q_x (x,y) \, dx \, dy = \int_C Q (x,y) \, dy. \quad (6-27)
\]

Adding Equations (6-26) and (6-27) gives us Equation (6-25), which completes the proof.
Figure 6.19  Integration over a standard region, where $C = C_3 + C_4$. 