Milne-Simpson Method

Another popular predictor-corrector scheme is known as the Milne-Simpson method. Its predictor is based on integration of \( f(t, y(t)) \) over the interval \([t_{k-3}, t_{k+1}]\):

\[
y(t_{k+1}) = y(t_{k-3}) + \int_{t_{k-3}}^{t_{k+1}} f(t, y(t)) \, dt.
\]

The predictor uses the Lagrange polynomial approximation for \( f(t, y(t)) \) based on the points \((t_{k-3}, f_{k-3}), (t_{k-2}, f_{k-2}), (t_{k-1}, f_{k-1}), \) and \((t_{k}, f_{k})\). It is integrated over the interval \([t_{k-3}, t_{k+1}]\). This produces the Milne predictor:

\[
p_{k+1} = y_{k-3} + \frac{4h}{3}(2f_{k-2} - f_{k-1} + 2f_{k}).
\]

The corrector is developed similarly. The value \( p_{k+1} \) can now be used. A second Lagrange polynomial for \( f(t, y(t)) \) is constructed, which is based on the points \((t_{k-1}, f_{k-1}), (t_{k}, f_{k}), \) and the new point \((t_{k+1}, f_{k+1}) = (t_{k+1}, f(t_{k+1}, p_{k+1}))\). The polynomial is integrated over \([t_{k-1}, t_{k+1}]\), and the result is the familiar Simpson’s rule:

\[
y_{k+1} = y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_{k} + f_{k+1}).
\]

Error Estimation and Correction

The error terms for the numerical integration formulas used to obtain both the predictor and corrector are of the order \(O(h^5)\). The L.T.E. for the formulas in (11) and (12) are

\[
y(t_{k+1}) - p_{k+1} = \frac{28}{90} y^{(5)}(c_{k+1})h^5 \quad \text{(L.T.E. for the predictor)},
\]

\[
y(t_{k+1}) - y_{k+1} = -\frac{1}{90} y^{(5)}(d_{k+1})h^5 \quad \text{(L.T.E. for the corrector)}.
\]

Suppose that \( h \) is small enough so that \( y^{(5)}(t) \) is nearly constant over the interval \([t_{k-3}, t_{k+1}]\). Then the terms involving the fifth derivative can be eliminated in (13) and (14) and the result is

\[
y(t_{k+1}) - p_{k+1} \approx \frac{28}{29}(y_{k+1} - p_{k+1}).
\]
Formula (15) gives an error estimate for the predictor that is based on the two computed values $p_{k+1}$ and $y_{k+1}$ and does not use $y^{(5)}(t)$. It can be used to improve the predicted value. Under the assumption that the difference between the predicted and corrected values at each step changes slowly, we can substitute $p_k$ and $y_k$ for $p_{k+1}$ and $y_{k+1}$ in (15) and get the following modifier:

\begin{equation}
    m_{k+1} = p_{k+1} + 28 \frac{y_k - p_k}{29}.
\end{equation}

This modified value is used in place of $p_{k+1}$ in the correction step, and equation (12) becomes

\begin{equation}
    y_{k+1} = y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_k + f(t_{k+1}, m_{k+1})).
\end{equation}

Therefore, the improved (modified) Milne-Simpson method is

\begin{align}
    p_{k+1} &= y_{k-3} + \frac{4h}{3}(2f_{k-2} - f_{k-1} + 2f_k) \quad \text{(predictor)}
    \\
    m_{k+1} &= p_{k+1} + 28 \frac{y_k - p_k}{29} \quad \text{(modifier)}
    \\
    f_{k+1} &= f(t_{k+1}, m_{k+1})
    \\
    y_{k+1} &= y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_k + f_{k+1}) \quad \text{(corrector)}.
\end{align}

Hamming’s method is another important method. We shall omit its derivation, but furnish a program at the end of the section. As a final precaution we mention that all the predictor-corrector methods have stability problems. Stability is an advanced topic and the serious reader should research this subject.

**Example 9.13.** Use the Adams-Bashforth-Moulton, Milne-Simpson, and Hamming methods with $h = \frac{1}{8}$ and compute approximations for the solution of the I.V.P.

\[y' = \frac{t - y}{2}, \quad y(0) = 1 \quad \text{over } [0, 3].\]

A Runge-Kutta method was used to obtain the starting values

\[y_1 = 0.94323919, \quad y_2 = 0.89749071, \quad \text{and} \quad y_3 = 0.86208736.\]

Then a computer implementation of Programs 9.6 through 9.8 produced the values in Table 9.12. The error for each entry in the table is given as a multiple of $10^{-8}$. In all entries there are at least six digits of accuracy. In this example, the best answers were produced by Hamming’s method.
Table 9.12  Comparison of the Adams-Bashforth-Moulton, Milne-Simpson, and Hamming Methods for Solving $y' = (t - y)/2$, $y(0) = 1$

<table>
<thead>
<tr>
<th>$k$</th>
<th>Adams-Bashforth-Moulton</th>
<th>Error</th>
<th>Milne-Simpson</th>
<th>Error</th>
<th>Hamming’s method</th>
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<td>1.00000000</td>
<td>$0E - 8$</td>
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<td>0.83640231</td>
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<td>0.81194555</td>
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