3.3 Upper-Triangular Linear Systems

We will now develop the back-substitution algorithm, which is useful for solving a linear system of equations that has an upper-triangular coefficient matrix. This algorithm will be incorporated in the algorithm for solving a general linear system in Section 3.4.

Definition 3.2. An $N \times N$ matrix $A = [a_{ij}]$ is called upper triangular provided that the elements satisfy $a_{ij} = 0$ whenever $i > j$. The $N \times N$ matrix $A = [a_{ij}]$ is called lower triangular provided that $a_{ij} = 0$ whenever $i < j$.

We will develop a method for constructing the solution to upper-triangular linear systems of equations and leave the investigation of lower-triangular systems to the reader. If $A$ is an upper-triangular matrix, then $AX = B$ is said to be an upper-
triangular system of linear equations and has the form

\[
\begin{align*}
 a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1N-1}x_{N-1} + a_{1N}x_N &= b_1 \\
 a_{22}x_2 + a_{23}x_3 + \cdots + a_{2N-1}x_{N-1} + a_{2N}x_N &= b_2 \\
 a_{33}x_3 + \cdots + a_{3N-1}x_{N-1} + a_{3N}x_N &= b_3 \\
 \vdots & \quad \vdots \\
 a_{N-1N}x_{N-1} + a_{N-1N}x_N &= b_{N-1} \\
 a_{NN}x_N &= b_N.
\end{align*}
\]

(1)

**Theorem 3.5 (Back Substitution).** Suppose that \( AX = B \) is an upper-triangular system with the form given in (1). If

\[
a_{kk} \neq 0 \quad \text{for} \quad k = 1, 2, \ldots, N,
\]

then there exists a unique solution to (1).

**Constructive Proof.** The solution is easy to find. The last equation involves only \( x_N \), so we solve it first:

\[
x_N = \frac{b_N}{a_{NN}}.
\]

Now \( x_N \) is known and it can be used in the next-to-last equation:

\[
x_{N-1} = \frac{b_{N-1} - a_{N-1N}x_N}{a_{N-1N-1}}.
\]

Now \( x_N \) and \( x_{N-1} \) are used to find \( x_{N-2} \):

\[
x_{N-2} = \frac{b_{N-2} - a_{N-2N-1}x_{N-1} - a_{N-2N}x_N}{a_{N-2N-2}}.
\]

Once the values \( x_N, x_{N-1}, \ldots, x_{k+1} \) are known, the general step is

\[
x_k = \frac{b_k - \sum_{j=k+1}^{N} a_{kj}x_j}{a_{kk}} \quad \text{for} \quad k = N-1, N-2, \ldots, 1.
\]

The uniqueness of the solution is easy to see. The \( N \)th equation implies that \( b_N/a_{NN} \) is the only possible value of \( x_N \). Then finite induction is used to establish that \( x_{N-1}, x_{N-2}, \ldots, x_1 \) are unique.

**Example 3.12.** Use back substitution to solve the linear system

\[
\begin{align*}
 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
 -2x_2 + 7x_3 - 4x_4 &= -7 \\
 6x_3 + 5x_4 &= 4 \\
 3x_4 &= 6.
\end{align*}
\]
Solving for $x_4$ in the last equation yields

$$x_4 = \frac{6}{3} = 2.$$ 

Using $x_4 = 2$ in the third equation, we obtain

$$x_3 = \frac{4 - 5(2)}{6} = -1.$$ 

Now $x_3 = -1$ and $x_4 = 2$ are used to find $x_2$ in the second equation:

$$x_2 = \frac{-7 - 7(-1) + 4(2)}{-2} = -4.$$ 

Finally, $x_1$ is obtained using the first equation:

$$x_1 = \frac{20 + 1(-4) - 2(-1) - 3(2)}{4} = 3.$$ 

The condition that $a_{kk} \neq 0$ is essential because equation (6) involves division by $a_{kk}$. If this requirement is not fulfilled, either no solution exists or infinitely many solutions exist.

**Example 3.13.** Show that there is no solution to the linear system

$$\begin{align*}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
0x_2 + 7x_3 - 4x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6.
\end{align*}$$

(7)

Using the last equation in (7), we must have $x_4 = 2$, which is substituted into the second and third equations to obtain

$$\begin{align*}
7x_3 - 8 &= -7 \\
6x_3 + 10 &= 4.
\end{align*}$$

The first equation in (8) implies that $x_3 = 1/7$, and the second equation implies that $x_3 = -1$. This contradiction leads to the conclusion that there is no solution to the linear system (7).

**Example 3.14.** Show that there are infinitely many solutions to

$$\begin{align*}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
0x_2 + 7x_3 + 0x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6.
\end{align*}$$

(9)
Using the last equation in (9), we must have \( x_4 = 2 \), which is substituted into the second and third equations to get \( x_3 = -1 \), which checks out in both equations. But only two values \( x_3 \) and \( x_4 \) have been obtained from the second through fourth equations, and when they are substituted into the first equation of (9), the result is

\[
(10) \quad x_2 = 4x_1 - 16,
\]

which has infinitely many solutions; hence (9) has infinitely many solutions. If we choose a value of \( x_1 \) in (10), then the value of \( x_2 \) is uniquely determined. For example, if we include the equation \( x_1 = 2 \) in the system (9), then from (10) we compute \( x_2 = -8 \).

Theorem 3.4 states that the linear system \( AX = B \), where \( A \) is an \( N \times N \) matrix, has a unique solution if and only if \( \det(A) \neq 0 \). The following theorem states that if any entry on the main diagonal of an upper- or lower-triangular matrix is zero, then \( \det(A) = 0 \). Thus, by inspecting the coefficient matrices in the previous three examples, it is clear that the system in Example 3.12 has a unique solution, and the systems in Examples 3.13 and 3.14 do not have unique solutions. The proof of Theorem 3.6 can be found in most introductory linear algebra textbooks.

**Theorem 3.6.** If the \( N \times N \) matrix \( A = [a_{ij}] \) is either upper or lower triangular, then

\[
(11) \quad \det(A) = a_{11}a_{22} \cdots a_{NN} = \prod_{i=1}^{N} a_{ii}.
\]

The value of the determinant for the coefficient matrix in Example 3.12 is \( \det A = 4(-2)(6)(3) = -144 \). The values of the determinants of the coefficient matrices in Examples 3.13 and 3.14 are both \( 4(0)(6)(3) = 0 \).