Theorem 12.24 (Convolution theorem) Let \( F(s) \) and \( G(s) \) denote the Laplace transforms of \( f(t) \) and \( g(t) \), respectively. Then the product given by \( H(s) = F(s)G(s) \) is the Laplace transform of the convolution of \( f \) and \( g \), is denoted \( h(t) = (f * g)(t) \), and has the integral representation

\[
h(t) = (f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau \quad \text{or} \quad (12-45)
\]

\[
h(t) = (g * f)(t) = \int_0^t g(\tau) f(t-\tau) d\tau. \quad (12-46)
\]

Proof The following proof is given for the special case when \( s \) is a real number. The general case is covered in advanced texts. Using the dummy variables \( \sigma \) and \( \tau \) and the integrals defining the transforms, we can express their product as

\[
F(s)G(s) = \left[ \int_0^\infty f(\sigma) e^{-s\sigma} d\sigma \right] \left[ \int_0^\infty g(\tau) e^{-st} d\tau \right].
\]

The product of integrals in this equation can be written as an iterated integral:

\[
F(s)G(s) = \int_0^\infty \left[ \int_0^\infty f(\tau) e^{-s(\sigma+\tau)} d\sigma \right] g(\tau) d\tau.
\]

We hold \( \tau \) fixed, use the change of variables \( t = \sigma + \tau \) and \( dt = d\sigma \), and rewrite the inner integral in the equation to obtain

\[
F(s)G(s) = \int_0^\infty \left[ \int_\tau^t f(t-\tau) e^{-st} dt \right] g(\tau) d\tau \\
= \int_0^\infty \left[ \int_\tau^t f(t-\tau) g(\tau) e^{-st} dt \right] d\tau.
\]

The region of integration for this last iterated integral is the wedge-shaped region in the \((t,\tau)\) plane shown in Figure 12.28. We reverse the order of integration in the integral to get

\[
F(s)G(s) = \int_0^\infty \left[ \int_0^t f(t-\tau) g(\tau) e^{-st} d\tau \right] dt.
\]

We rewrite this equation as

\[
F(s)G(s) = \int_0^\infty \left[ \int_0^t f(t-\tau) g(\tau) d\tau \right] e^{-st} dt \\
= \mathcal{L}^{-1} \left( \int_0^t f(t-\tau) g(\tau) d\tau \right).
\]

which establishes Equation (12-46). We can interchange the role of the functions \( f(t) \) and \( g(t) \), so Equation (12-45) follows immediately.
Figure 12.28 The region of integration in the convolution theorem.

Table 12.4 lists the properties of convolution.

<table>
<thead>
<tr>
<th>Property</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutative</td>
<td>$f * g = g * f$</td>
</tr>
<tr>
<td>Distributive</td>
<td>$f * (g + h) = f * g + f * h$</td>
</tr>
<tr>
<td>Associative</td>
<td>$(f * g) * h = f * (g * h)$</td>
</tr>
<tr>
<td>Zero</td>
<td>$f * 0 = 0$</td>
</tr>
</tbody>
</table>

Table 12.4 Properties of Convolution