Abstract

We prove some new results on Hadwin’s general version of reflexivity that reduce the study of $E$-reflexivity (or $E$-hyperreflexivity) of a linear subspace to a smaller linear subspace. By applying our abstract results, we present a simple proof of D. Hadwin’s theorem, which states that every $C^*$-algebra is approximately hyperreflexive. We also prove that the image of any $C^*$-algebra under any bounded unital homomorphism into the operators on a Banach space is approximately reflexive. We introduce a new version of reflexivity, called approximate algebraic reflexivity, and study its properties.

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1. Introduction and preliminaries

Reflexivity was introduced by P.R. Halmos [12] for lattices of (closed) subspace of a Hilbert Space $\mathcal{H}$, and a subalgebra of $B(\mathcal{H})$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat(\mathcal{A})$, where $AlgLat(\mathcal{A})$ is the collection of all operators in $B(\mathcal{H})$ that leave invariant every $\mathcal{A}$-invariant subspace of $\mathcal{H}$. This notion was extended by A. Loginov and V. Shulman [16] to linear subspaces of $B(\mathcal{H})$. Algebraic reflexivity was introduced by D. Hadwin [4] for subspaces of $L(V)$, where $L(V)$ is the algebra of all linear transformations on a vector space $V$ over a field $\mathbb{F}$. Many important results on algebraic reflexivity were obtained by D. Larson [15]. Based on K.J. Harrison’s notion of strongly reductive operators [13], C. Apostol, C. Foiaş, and D. Voiculescu [1] introduced the notion of approximate reflexivity for a subalgebra $\mathcal{A}$ of $B(\mathcal{H})$. This notion of reflexivity was generalized to subspaces by D. Hadwin [5]. In [7], D. Hadwin unified most of these notions.
of reflexivity into his general version of reflexivity. He proved several results in this setting, surprisingly, with elementary methods. He obtained many important results (some known and some unknown) as simple corollaries to his results in this setting.

In this paper we follow D. Hadwin’s steps and obtain some new results in his general version of reflexivity. In Section 2 we obtain some abstract results in the general setting and we apply them to derive some results in Sections 3 and 4. In Section 4 we introduce a new version of algebraic reflexivity, called approximate algebraic reflexivity, that has many interesting properties.

Throughout the paper we use $\mathcal{H}$ to denote a Hilbert space over the field of complex numbers $\mathbb{C}$, and $\mathcal{B}(\mathcal{H})$ denotes the Banach algebra of all bounded linear transformations (operators) on $\mathcal{H}$. We let $\mathcal{M}_n(\mathbb{C})$ denote the space of $n \times n$ matrices over $\mathbb{C}$, i.e., $\mathcal{M}_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$. In general, if $X$ is a Banach space, $\mathcal{B}(X)$ denotes the space of all bounded linear transformations on $X$, and $\mathcal{K}(X)$ denotes the algebra of all compact operators in $\mathcal{B}(X)$. Here $V$ denotes a vector space over a field $\mathbb{F}$, and we let $\mathcal{L}(V)$ denote the set of $\mathbb{F}$-linear transformations on $V$. We let $\mathcal{F}(V)$ denote the space of all finite rank transformations on $V$. If $W$ is a Banach space, $W^\#$ denotes the norm dual of $W$ and $V^\perp$ denotes the space of all $\mathbb{F}$-linear maps from $V$ into $\mathbb{F}$. For $x \in V$ and $\alpha \in V^\perp$, the rank-one tensor $x \otimes \alpha$ is the linear map defined on $\mathcal{L}(V)$ by $(x \otimes \alpha)(S) = \alpha(Sx)$ for every $S \in \mathcal{L}(V)$.

A linear subspace $S$ of $\mathcal{L}(V)$ is called algebraically reflexive if and only if $S = \text{Ref}_0(S)$, where

$$\text{Ref}_0(S) = \{ T \in \mathcal{L}(V): Tx \in Sx, \forall x \in V \}.$$

For an algebra $A \subset \mathcal{B}(\mathcal{H})$, we say:

$$T \in \text{ApprAlgLat}(A)$$

if and only if $\|(1 - P_\lambda)TP_\lambda\| \to 0$ for every net $\{P_\lambda\}$ of projections in $\mathcal{B}(\mathcal{H})$ for which $\|(1 - P_\lambda)AP_\lambda\| \to 0$, $\forall A \in A$. We say that $A$ is approximately reflexive if $A = \text{ApprAlgLat}(A)$.

The notion of approximate reflexivity was generalized to include subspaces by D. Hadwin [5]. For a subspace $S \subset \mathcal{B}(\mathcal{H})$, we say:

$$T \in \text{ApprRef}(S)$$

if and only if $\|P_\lambda TQ_\lambda\| \to 0$ for all nets of projections $\{P_\lambda\}$ and $\{Q_\lambda\}$ for which $\|P_\lambda SQ_\lambda\| \to 0$, $\forall S \in S$. We say $S$ is approximately reflexive if $S = \text{ApprRef}(S)$.

It is not trivial that $\text{ApprAlgLat}(A) = \text{ApprRef}(A)$ when $A$ is a unital algebra, but it is indeed true [5].

D. Hadwin defined the notion of approximate hyperreflexivity for an approximately reflexive unital algebra $A$ of $\mathcal{B}(\mathcal{H})$. We define the seminorm $d_a(\cdot, A)$ on $\mathcal{B}(\mathcal{H})$ by

$$d_a(T, A) = \sup \limsup \lambda \|(1 - P_\lambda)TP_\lambda\|,$$

where the supremum is taken over all nets $\{P_\lambda\}$ of projections in $\mathcal{B}(\mathcal{H})$ such that $\|(1 - P_\lambda)AP_\lambda\| \to 0$ for every $A \in A$. The smallest $K \geq 1$ for which

$$\text{dist}(T, A) \leq Kd_a(T, A)$$
for all \( T \in \mathcal{B}(\mathcal{H}) \) is the constant of approximate hyperreflexivity of \( \mathcal{A} \), and is denoted by \( K_a(\mathcal{A}) \). D. Hadwin [5] proved that \( K_a(\mathcal{A}) \leq 29 \) whenever \( \mathcal{A} \) is a unital C*-subalgebra of \( \mathcal{B}(\mathcal{H}) \).

For the sake of completeness, we recall D. Hadwin’s general version of reflexivity that contains the usual topological, approximate, and algebraic versions of reflexivity as special cases [7].

Suppose \( \mathbb{F} \) is a topological field with Hausdorff topology, \( X \) is a vector space over \( \mathbb{F} \), and \( Y \) is a vector space of linear maps from \( X \) to \( \mathbb{F} \) that separates the points of \( X \) (i.e., \( \bigcap_{f \in Y} \ker(f) = \{0\} \)). Such a pair \( (X, Y) \) is called a dual pair over \( \mathbb{F} \). We define \( (X, Y, E) \) to be a reflexivity triple over \( \mathbb{F} \) if \( (X, Y) \) is a dual pair over \( \mathbb{F} \), \( E \subset Y \) is closed under multiplication by scalars, and \( E_\perp := \bigcap_{\varphi \in E} \ker(\varphi) = \{0\} \). If \( S \) is a linear subspace of \( X \), we define \( \text{Ref}_E(S) = (S_\perp \cap E)_\perp \), where

\[
S_\perp = \{ \varphi \in Y : \varphi(S) = 0, \forall S \in S \}.
\]

We say that \( S \) is \( E \)-reflexive if \( S = \text{Ref}_E(S) \).

We define the \( \sigma(X, Y) \)-topology on \( X \) to be the smallest topology on \( X \) that makes all the maps in \( Y \) continuous. For \( S \subset X \), \( \overline{S} \) denotes the \( \sigma(X, Y) \)-closure of \( S \) in \( X \), and \( \overline{\text{span}}(S) \) denotes the \( \sigma(X, Y) \)-closed linear span of \( S \).

Next we recall D. Hadwin’s notion of hyperreflexivity. Here \( \mathbb{F} \) is the field of complex or real numbers and \( X \) and \( Y \) are (real or complex) normed spaces, with \( Y \) a subspace of the normed dual \( X^\# \) of \( X \). In this case we call \( (X, Y, E) \) a normed reflexivity triple. If \( (X, Y, E) \) is a normed reflexivity triple and \( S \) is a linear subspace of \( X \), we define a seminorm \( d_Y(\cdot, S) \) on \( X \) by

\[
d_Y(x, S) = \sup\{||f(x)|| : f \in S_\perp, ||f|| = 1\}.
\]

We define another seminorm \( d_E(\cdot, S) \) on \( X \) by

\[
d_E(x, S) = \sup\{||f(x)|| : f \in S_\perp \cap E, ||f|| = 1\}.
\]

It is clear that \( d_E(x, S) \leq d_Y(x, S) \) and that \( x \in \text{Ref}_E(S) \) if and only if \( d_E(x, S) = 0 \). We say that \( S \) is \( E \)-hyperreflexive if there is a nonnegative constant \( K \) such that \( d_Y(x, S) \leq K d_E(x, S) \) for every \( x \in X \); the smallest such \( K \geq 1 \) is called the constant of \( E \)-hyperreflexivity of \( S \) and is denoted by \( K_E(S) \). It is clear that if \( S \) is a closed subspace of \( X \), then \( E \)-hyperreflexivity of \( S \) implies \( E \)-reflexivity of \( S \).

We next recall the notions of relative reflexivity and hyperreflexivity. Suppose that \( (X, Y, E) \) is a reflexivity triple and \( M \) is a subspace of \( X \). A subspace \( S \) of \( M \) is \( E \)-reflexive relative to \( M \) if \( \text{Ref}_E(S) \cap M = S \). Note that \( (M, Y/M_\perp, E/M_\perp) \) is a reflexivity triple, and \( S \) is \( E \)-reflexive with respect to \( M \) if and only if \( S \) is \( E/M_\perp \)-reflexive.

Next suppose that \( (X, Y, E) \) is a normed reflexivity triple and \( M \) is a subspace of \( X \). A subspace \( S \) of \( M \) is \( E \)-hyperreflexive relative to \( M \) if there is a smallest number \( K = K_E(S, M) \) such that \( d_Y(x, S) \leq K d_E(x, S) \) for every \( x \in M \).

The connections between the general version of reflexivity and some of the classical versions of reflexivity can be found in [7].
2. A few results in the general setting

Theorems 3 and 6 are the main results of this section. These two theorems, basically, provide tools to reduce the problem of $E$-reflexivity (or $E$-hyperreflexivity) of a subspace of $X$ to a smaller subspace of $X$. We also introduce a general version of approximate $E$-reflexivity.

Throughout this section, $(X, Y, E)$ will be a reflexivity triple or a normed reflexivity triple. Suppose $F$ is a subspace of $X$ for which $F \perp \subset E$. Then it follows that

$$Ref_E(F) = (F \perp \cap E)_{\perp} = F.$$ 

Therefore if $F$ is closed, then it is $E$-reflexive. The next proposition states a generalization of this.

**Proposition 1.** Suppose $F$ and $S$ are subspaces (not necessarily closed) of $X$ such that $F \perp \subset E$. Then $S + F$ is $E$-reflexive whenever $S + F$ is closed in $X$. In particular, any closed subspace $T$ containing $F$ is $E$-reflexive.

**Proof.** Since $(S + F)_{\perp} \subset F_{\perp} \subset E$, then

$$Ref_E(S + F) = ((S + F)_{\perp} \cap E)_{\perp} = S + F.$$ 

A hyperreflexivity version of the preceding proposition can be proved for normed reflexivity triples.

**Proposition 2.** Suppose $(X, Y, E)$ is a normed reflexivity triple, $S$ and $F$ are subspaces (not necessarily closed) of $X$ such that $F \perp \subset E$. Then

1. $S + F$ is $E$-hyperreflexive when $S + F$ is closed. In particular, any closed linear subspace $S$ of $X$ containing $F$ is $E$-hyperreflexive with $K_E(S) = 1$.
2. If $S$ is closed, then, $S$ is $E$-hyperreflexive if and only if there is a constant $K$ such that $d_Y(e, S) \leq K d_E(e, S)$, for every $e \in F$.

**Proof.** (1) Since $(S + F)_{\perp} \subset F_{\perp} \subset E$, it follows that

$$d_Y(x, S + F) = \sup \{ ||f(x)|| : f \in (S + F)_{\perp}, ||f|| = 1 \}$$

$$= \sup \{ ||f(x)|| : f \in (S + F)_{\perp} \cap E, ||f|| = 1 \}$$

$$= d_E(x, S + F).$$

The above equality shows that $S + F$ is $E$-hyperreflexive with the constant of $E$-hyperreflexivity equal to 1.

(2) Suppose that $d_Y(e, S) \leq K d_E(e, S)$ for every $e \in F$. Then for every $e \in F$ and $s \in S$ we have:

$$d_Y(e + s, S) = d_Y(e, S) \leq K d_E(e, S) = K d_E(e + s, S).$$
Since the seminorms $d_Y(\cdot, \mathcal{S})$ and $d_E(\cdot, \mathcal{S})$ are norm continuous, we conclude that $d_Y(x, \mathcal{S}) \leq K d_E(x, \mathcal{S})$ for every $x \in \mathcal{S} + \mathcal{F}$. Thus $\mathcal{S}$ is $E$-hyperreflexive in $\mathcal{S} + \mathcal{F}$. However, by part (1), $\mathcal{S} + \mathcal{F}$ is $E$-hyperreflexive in $X$. Hence $\mathcal{S}$ is $E$-hyperreflexive in $X$, by [7, Theorem 2.6]. □

The existence of a subspace $\mathcal{F}$ of $X$ as in the previous proposition makes it much easier to work with $E$-reflexivity. Suppose that $\mathcal{F}$ is a closed subspace of $X$ such that $\mathcal{F}^\perp \subset E$, and let $\mathcal{S}$ be an $E$-reflexive subspace of $X$. Then it follows that $\text{Ref}_E(\mathcal{S} \cap \mathcal{F}) = \mathcal{S} \cap \mathcal{F}$. In other words the $E$-reflexivity of $\mathcal{S}$ implies the $E$-reflexivity of $\mathcal{S} \cap \mathcal{F}$. It turns out that with some extra assumptions the converse is also true. The next theorem says more.

**Theorem 3.** Suppose $\mathcal{F}$, $\mathcal{S}$, and $\mathcal{F} + \mathcal{S}$ are closed subspaces of $X$ and $\mathcal{F}^\perp + E = E$. Then the following are true whenever the natural mapping $\eta: (\mathcal{S} + \mathcal{F})/\mathcal{S} \to \mathcal{F}/(\mathcal{S} \cap \mathcal{F})$ defined by $\eta(s + e + \mathcal{S}) = e + \mathcal{S} \cap \mathcal{F}$, for every $s \in \mathcal{S}$ and $e \in \mathcal{F}$, is continuous:

1. $\text{Ref}_E(\mathcal{S} \cap \mathcal{F}) = \text{Ref}_E(\mathcal{S} \cap \mathcal{F})$.
2. $\text{Ref}_E(\mathcal{S}) = \mathcal{S} + \text{Ref}_E(\mathcal{S} \cap \mathcal{F})$.
3. $\mathcal{S}$ is $E$-reflexive if and only if $\mathcal{S} \cap \mathcal{F}$ is $E$-reflexive.

**Proof.** (1) It is clear that $\text{Ref}_E(\mathcal{S} \cap \mathcal{F}) \subset \text{Ref}_E(\mathcal{S} \cap \mathcal{F})$. Thus we just need to show the other inclusion, $\text{Ref}_E(\mathcal{S} \cap \mathcal{F}) \subset \text{Ref}_E(\mathcal{S} \cap \mathcal{F})$. If this is not true, then 

$$\exists x \in (\text{Ref}_E(\mathcal{S} \cap \mathcal{F}) \setminus \text{Ref}_E(\mathcal{S} \cap \mathcal{F})) \quad \text{and} \quad \exists \varphi \in E \quad \text{such that} \quad \varphi(x) = 1 \quad \text{and} \quad \varphi(T) = 0, \quad \forall T \in \mathcal{S} \cap \mathcal{F}.$$ 

Let $g: \mathcal{S} + \mathcal{F} \to (\mathcal{S} + \mathcal{F})/\mathcal{S}$ be the quotient map and define 

$$\varphi': \mathcal{F}/(\mathcal{S} \cap \mathcal{F}) \to \mathbb{C} \quad \text{by} \quad \varphi'(e + \mathcal{S} \cap \mathcal{F}) = \varphi(e), \quad \forall e \in \mathcal{F}.$$ 

Note that $\varphi'$ is well defined and continuous, because 

$$\ker(\varphi') = \ker(\varphi) + \mathcal{S} \cap \mathcal{F} = \ker(\varphi)$$

is a closed subspace. Define $\psi: \mathcal{S} + \mathcal{F} \to \mathbb{C}$ by 

$$\psi(s + e) = \varphi(e), \quad \forall e \in \mathcal{F} \quad \text{and} \quad \forall s \in \mathcal{S}.$$ 

It is clear that $\psi$ is well defined. It is easy to check that 

$$\psi = \varphi' \circ \eta \circ g \quad \text{and} \quad \ker(\psi) = \mathcal{S} + \ker(\varphi).$$ 

This implies that $\psi$ is continuous whenever $\eta$ is continuous. Therefore $\psi \in Y$. In fact, since $\psi - \varphi$ annihilates $\mathcal{F}$, we have 

$$\psi = (\psi - \varphi) + \varphi \in \mathcal{F}^\perp + E \subset E.$$ 

The fact that $\psi(x) = \varphi(x) = 1$ and $\psi \in E$ annihilates $\mathcal{S}$, violates the assumption that $x \in \text{Ref}_E(\mathcal{S})$. This completes the proof of the first statement.
(2) Clearly \( S + \text{Ref}_E(S \cap F) \subset \text{Ref}_E(S) \). On the other hand, suppose \( z \in \text{Ref}_E(S) \subset F + S \). Thus \( z = s + e \) for some \( s \in S \) and \( e \in F \). By part (1), it follows that
\[
z - s = e \in \text{Ref}_E(S) \cap F = \text{Ref}_E(S \cap F).
\]
Therefore \( z \in S + \text{Ref}_E(S \cap F) \).

(3) If \( S \) is \( E \)-reflexive, then clearly, by part (1), \( S \cap F \) is \( E \)-reflexive and if \( S \cap F \) is \( E \)-reflexive, then clearly by (2) \( S \) is \( E \)-reflexive. □

**Remark 4.**

(1) Note that the map \( \eta \) in Theorem 3 is continuous when \( S \) is finite-dimensional. Moreover, if all spaces are Banach spaces, then the Closed-Graph theorem implies that the map \( \eta \) is continuous.

(2) It can be easily shown that the statements (1) and (2) in Theorem 3 are equivalent without the continuity condition on the map \( \eta \).

Several applications of Theorem 3 will be provided in the next sections. Next we will present an \( E \)-hyperreflexive analogous of Theorem 3. For the case that \( X \) is a Banach space, we find an estimate for the norm of the map \( \eta \) defined in Theorem 3 in terms of the norm of another map whose norm is easier to compute in a lot of cases. This estimate can be used to find an estimate on the constant of \( E \)-hyperreflexivity of subspaces of \( X \).

**Lemma 5.** Suppose \( X \) is a Banach space and \( F, S, \) and \( F + S \) are closed subspaces of \( X \). Let
\[
\eta: \frac{S + F}{S} \to \frac{F}{S \cap F} \quad \text{and} \quad \beta: \frac{S + F}{F} \to \frac{S}{S \cap F}
\]
be the natural maps defined by
\[
\eta(s + e + S) = e + S \cap F \quad \text{and} \quad \beta(s + e + F) = s + S \cap F
\]
for every \( s \in S \) and \( e \in F \). Then \( \|\eta\| \leq 1 + \|\beta\| \).

**Proof.** We need to show that
\[
dist(e, S \cap F) \leq (1 + \|\beta\|) dist(e, S), \quad \forall e \in F.
\]
Let \( e \in F \). Then there exists a sequence \( \{s_n\} \) in \( S \) such that \( dist(e, S) = \lim_{n \to \infty} \|e - s_n\| \). For every \( n \geq 1 \), choose \( y_n \in S \cap F \) such that \( dist(s_n, S \cap F) = \|s_n - y_n\| - 1/n \). Hence for every \( n \geq 1 \) we have:
\[
dist(e, S) \geq dist(s_n, F) = \|s_n + F\| \geq \frac{1}{\|\beta\|} \|\beta(s_n + F)\|
\]
\[
= \frac{1}{\|\beta\|} \|s_n + S \cap F\| = \frac{1}{\|\beta\|} dist(s_n, S \cap F)
\]
\[
= \frac{1}{\|\beta\|} \|s_n - y_n\| - \frac{1}{n\|\beta\|}.
\]
Therefore we have
\[
\text{dist}(e, S \cap \mathcal{F}) \leq \|e - y_n\| \leq \|e - s_n\| + \|s_n - y_n\| \\
\leq \|e - s_n\| + \|\beta\| \left(\|e - s_n\| + \frac{1}{n}\right).
\]

By letting \(n \to \infty\) we get what we desired. \(\square\)

**Theorem 6.** Suppose \(X\) is a Banach space and \((X, Y, E)\) is a normed reflexivity triple. Suppose also that \(\mathcal{F}, S,\) and \(S + \mathcal{F}\) are closed subspaces of \(X\) such that \(\mathcal{F}^\perp + E = E\). Then \(S\) is \(E\)-hyperreflexive if and only if \(S \cap \mathcal{F}\) is \(E\)-hyperreflexive.

**Proof.** First suppose that \(S\) is \(E\)-hyperreflexive in \(X\). Since \(\mathcal{F} \subset E\), we have
\[
d_Y(x, \mathcal{F}) = d_E(x, \mathcal{F}) = \text{dist}(x, \mathcal{F}), \quad \forall x \in X.
\]

Therefore by [7, Theorem 2.8] we can conclude that \(S \cap \mathcal{F}\) is \(E\)-hyperreflexive.

To prove the other direction, suppose \(S \cap \mathcal{F}\) is \(E\)-hyperreflexive. If \(\mathcal{F} = S \cap \mathcal{F}\), then \(\mathcal{F} \subset S\), hence \(S\) is \(E\)-hyperreflexive by Proposition 2. Thus we can assume that \(\mathcal{F} \neq S \cap \mathcal{F}\). We first show that \(S\) is \(E\)-hyperreflexive relative to \(S + \mathcal{F}\). To do this, suppose that \(z = s_1 + e_1 \in S + \mathcal{F}\), where \(s_1 \in S\) and \(e_1 \in \mathcal{F}\) and let \(K > K_E(S \cap \mathcal{F})\). If \(d_Y(z, S) = 0\), then \(d_E(z, S) \geq ld_Y(z, S)\) for every number \(l\). Assume that \(d_Y(z, S) \neq 0\). Hence \(d_Y(e_1, S \cap \mathcal{F}) \neq 0\) and we have
\[
d_Y(e_1, S \cap \mathcal{F}) < Kd_E(e_1, S \cap \mathcal{F}).
\]

Therefore there exists a function \(f = f_{e_1} \in (S \cap \mathcal{F})^\perp \cap E\) such that
\[
\|f\| = 1 \quad \text{and} \quad d_Y(e_1, S \cap \mathcal{F}) < K|f(e_1)|.
\]

Let
\[
\gamma : S + \mathcal{F} \to \frac{S + \mathcal{F}}{S}
\]
be the quotient map and define the maps
\[
\eta : \frac{S + \mathcal{F}}{S} \to \frac{\mathcal{F}}{S \cap \mathcal{F}} \quad \text{and} \quad \tilde{f} : \frac{\mathcal{F}}{S \cap \mathcal{F}} \to \mathbb{C}
\]
by
\[
\eta(s + e + S) = e + S \cap \mathcal{F}, \quad \forall s \in S, \forall e \in \mathcal{F},
\]
\[
\tilde{f}(e + S \cap \mathcal{F}) = f(e), \quad \forall e \in \mathcal{F}.
\]

The linear map \(\eta\) is continuous by Remark 4, and \(\tilde{f}\) is continuous because
\[
\ker(\tilde{f}) = S \cap \mathcal{F} + \ker(f) = \ker(f)
\]
is a closed subspace. Therefore the map $g : S + F \to \mathbb{C}$ defined by $g(s + e) = f(e)$, for every $e \in F$, is continuous (because $g = \tilde{f} \circ \eta \circ \gamma$). In fact, $g \in S^\perp$, $g|_F = f|_F$ and

$$\|g\| \leq \|\tilde{f}\| \|\eta\| \|\gamma\| \leq \|f\| \|\eta\| \leq \|\eta\|.$$ 

Therefore

$$g = (g - f) + f \in F^\perp + E = E.$$ 

Hence $g/\|g\| \in S^\perp \cap E$ and it follows that

$$d_E(z, S) = d_E(s_1 + e_1, S) = d_E(e_1, S) \geq \frac{|g(e_1)|}{\|g\|} = \frac{|f(e_1)|}{\|g\|} \geq \frac{d_Y(e_1, S \cap F)}{K \|\eta\|} \geq \frac{d_Y(e_1, S)}{K \|\eta\|} = \frac{d_Y(z, S)}{K \|\eta\|}.$$ 

Thus we have

$$d_Y(z, S) \leq K \|\eta\|d_E(z, S), \quad \forall z \in S + F.$$ 

Therefore $S$ is $E$-hyperreflexive in $S + F$.

Finally, since by Proposition 2, $S + F$ is $E$-hyperreflexive in $X$ with the constant of $E$-hyperreflexivity equal to $1$, by [7, Theorem 2.6], $S$ is $E$-hyperreflexive in $X$ and

$$K_E(S) \leq (K \|\eta\| + 1)(1 + 1) - 1 = 2K \|\eta\| + 1. \quad \square$$ 

In view of the fact that in the proof of Theorem 6, the number $K > K_E(S \cap F)$ was arbitrary, we obtain the following corollary.

**Corollary 7.** Under the assumptions of Theorem 6

$$K_E(S) \leq 2K_E(S \cap F)\|\eta\| + 1 \leq 2(1 + \|\beta\|)K_E(S \cap F) + 1,$$

where $\beta$ and $\eta$ are the maps defined in Lemma 5.

We next consider a general notion of approximate reflexivity. Suppose $(X, Y, E)$ is a reflexivity triple, and let $\bar{E}$ denote the $\sigma(Y, X)$-closure of $E$. Then $(X, Y, \bar{E})$ is also a reflexivity triple. We define *approximate $E$-reflexivity* to be $\bar{E}$-reflexivity, and we define

$$\text{ApprRef}_E(S) = \text{Ref}_{\bar{E}}(S).$$

Suppose $F \subset X$ is a linear subspace such that $F^\perp + \bar{E} = \bar{E}$. Then the results of this section apply to approximate reflexivity. In particular, if $S + F$ is $\sigma(X, Y)$-closed, then $\text{ApprRef}_E(S) = S + \text{ApprRef}_E(S \cap F)$. In some cases, we can further describe $\text{ApprRef}_E(S \cap F)$.

**Theorem 8.** Suppose that $(X, Y, E)$ is a reflexivity triple, $F$ is a closed subspace of $X$, $S$ is a subspace of $F$, $E_0 \subset E$ is closed under scalar multiplication, and $E_0 + F^\perp = E$. Then $\text{Ref}_E(S) = \text{Ref}_{E_0}(S \cap F) \cap F$. 


Proof. Since \( S^\perp \cap E_0 \subseteq S^\perp \cap E \), it follows that \( Ref_E(S) \subseteq Ref_{E_0}(S) \cap F \). If the reverse inclusion is not valid, then there are \( x \in Ref_{E_0}(S) \cap F \setminus Ref_E(S) \) and \( \varphi \in E \) such that \( \varphi|_S = 0 \) and \( \varphi(x) = 1 \). The equality \( E_0 + F^\perp = E \) implies that \( \varphi = \psi + \eta \) for some \( \psi \in E_0 \) and \( \eta \in F^\perp \subseteq S^\perp \). Therefore \( \psi \) annihilates \( S \) and \( \psi(x) = \varphi(x) - \eta(x) = 1 \). This contradicts the fact that \( x \in Ref_{E_0}(S) \). \( \square \)

**Corollary 9.** Suppose that \((X, Y, E)\) is a reflexivity triple, \( F \) is a closed subspace of \( X \), \( E + F^\perp = \overline{E} \), and \( S \) is a subspace of \( F \). Then \( ApprRef_E(S) = F \cap Ref_E(S) \).

The last corollary shows how we can sometimes relate approximate \( E \)-reflexivity with \( E \)-reflexivity.

**Corollary 10.** Suppose that \((X, Y, E)\) is a reflexivity triple, \( F \) is a closed subspace of \( X \), \( E + F^\perp = \overline{E} \), and \( S \) is a subspace of \( X \) such that \( S + F \) is closed. Then \( S \) is approximately \( E \)-reflexive if and only if \( F \cap Ref_E(S \cap F) = S \cap F \).

3. Applications to approximate reflexivity and hyperreflexivity

In this section the results of the previous section will be applied to approximate reflexivity and approximate hyperreflexivity. In the Banach space setting we slightly modify D. Hadwin’s definition of approximate reflexivity [7] and [5]. This new definition is the same when the Banach space has certain properties, in particular when the Banach space is a Hilbert space. The benefit of the modification is that we can easily apply the results of the preceding section to prove many new interesting results. Our main result in this section is Theorem 21, which says that the image of any \( C^* \)-algebra under any bounded unital homomorphism into \( B(\mathcal{W}) \) is approximately reflexive, where \( \mathcal{W} \) is a Banach space.

Throughout this section \( \mathcal{W} \) is a Banach space, by \( \overline{F(\mathcal{W})} \) we mean the norm-closure of all finite-rank operators on \( \mathcal{W} \).

If in the reflexivity triple \((X, Y, E)\), we let \( X = B(\mathcal{W}) \), \( Y = B(\mathcal{W})^\# \) (normed dual), and

\[
E = \{ x \otimes \alpha \colon x \in \mathcal{W}, \alpha \in \mathcal{W}^\# \}_{\text{weak}^*} \subset Y,
\]

then we define \( ApprRef(S) = Ref_E(S) \) and we say that \( S \) is *approximately reflexive* if and only if \( ApprRef(S) = S \). Note that our definition of \( ApprRef(S) \) differs slightly from that in [5,7], where \( E \) is defined to be the weak* limits of bounded nets of rank-one tensors. The advantage of our definition is because of the alternative characterization given in Lemma 14.

Suppose \( S \) is a linear space of linear transformations on \( V \). A linear map \( \varphi : S \to \mathbb{F} \) is said to be *completely rank-nonincreasing* if

\[
rank(\varphi(S_{i,j})) \leq rank((S_{i,j})), \quad \forall n \geq 1 \text{ and } \forall (S_{i,j}) \in \mathcal{M}_n(S).
\]

Please see [8,11,14] for some results on completely rank-nonincreasing maps.

The following lemma is due to Larson [15].

**Lemma 11.** Let \( \mathcal{R} \) be a finite-dimensional subspace of \( \mathcal{L}(V) \) with \( \mathcal{R} \cap \mathcal{F}(V) = \{0\} \). Then if \( W_1 \) and \( W_2 \) are linear subspaces of \( V \) of finite codimension, there exists a vector \( x \in W_1 \) which is separating for \( \mathcal{R} \) such that \( \mathcal{R}(x) \subseteq W_2 \).
By using Lemma 11, D. Larson showed in [15] that if $S$ is a finite-dimensional subspace of $\mathcal{L}(V)$, then $S$ is algebraic reflexive if and only if $S \cap \mathcal{F}(V)$ is algebraic reflexive.

**Corollary 12.** Suppose $\varphi : \mathcal{L}(V) \to \mathbb{F}$ is completely rank nonincreasing. Then for every finite subset $S$ of $\mathcal{L}(V)$, $\varphi|_S$ can be represented by a rank-one tensor.

**Proof.** Suppose that $S$ is a finite subset of $\mathcal{L}(V)$. We will show that $\varphi$ restricted to the linear subspace generated by $S$ is a rank-one tensor. Thus we can assume that $S$ is a finite-dimensional vector space. The subspaces $S$ can be written as $S = S_F + T$, where $S_F = S \cap \mathcal{F}(V)$ and $S_F \cap T = \{0\}$. Hence there exists a finite rank idempotent $P \in \mathcal{L}(V)$ such that $PSP = S$ for every $S \in S_F$. Since $S_F \subset P \mathcal{L}(V)P \cong \mathcal{M}(\mathbb{F})$ for some $n$, by using the fact (see [8]) that every completely rank-nonincreasing map on $\mathcal{M}(\mathbb{F})$ can be represented as a rank-one tensor, it follows that $\varphi|_{S_F} = x_1 \otimes \alpha$ for some $x_1 \in \text{ran}(P)$ and some linear map $\alpha$ on $\text{ran}(P)$. Let $W_1 = W_2$ be the (algebraic) complement of the linear subspace generated by $S(x_1)$. Then by Lemma 11, there exists a vector $x_2 \in W_1$ which is a separating vector for $T$ and $T(x_2) \subset W_1$. Since $x = x_1 + x_2$ is also a separating vector for $T$ and $S(x) \cap T(x) = \{0\}$, we can extend the map $\alpha$ linearly to $S(x)$ by defining $\alpha(Tx) = \varphi(T)$ for every $T \in T$. Therefore $\varphi|_S = x \otimes \alpha$ is a rank-one tensor. 

Note that if in Corollary 12, the map $\varphi$ is defined on $B(W)$ and is continuous, then the map $\alpha$ can be extended to $W$ continuously.

**Corollary 13.** Let $\varphi : \mathcal{L}(V) \to \mathbb{F}$ be a linear map. The following are equivalent:

1. $\varphi$ is completely rank-nonincreasing.
2. $\varphi$ is a limit of rank-one tensors.

**Proof.** Suppose $\varphi$ is a completely rank-nonincreasing. Let

$$
\Lambda = \{ \lambda : \lambda \text{ is a finite linearly independent subset of } \mathcal{L}(V) \}.
$$

For each $\lambda \in \Lambda$, let $F_\lambda$ be the linear space generated by $\lambda$. By Corollary 12, $\varphi|_{F_\lambda} = x_\lambda \otimes \alpha_\lambda$, for some $x_\lambda \in V$ and a linear map $\alpha_\lambda$ defined on $V$. It is clear that $\varphi(T) = \lim_\lambda (x_\lambda \otimes \alpha_\lambda)(T)$, for every $T \in \mathcal{L}(V)$. The other direction is easy, since every rank-one tensor is completely rank-nonincreasing.

A Banach space $\mathcal{W}$ is said to be a Glimm space if, for each $\phi \in B(\mathcal{W})^\#$ that annihilates $\mathcal{K}(\mathcal{W})$ there is a bounded net $\{x_\lambda\}$ converging weakly to 0 in $\mathcal{W}$ and a bounded net $\{\alpha_\lambda\}$ converging weak* to 0 in $\mathcal{W}^\#$ such that $\phi(T) = \lim_\lambda (x_\lambda \otimes \alpha_\lambda)T$ for every $T \in B(\mathcal{W})$. In [6], D. Hadwin showed that if $\mathcal{W}$ is $c_0$, a Hilbert space, $\ell_p$ $(1 < p < \infty)$, or the set of trace-class operators on a Hilbert space, then $\mathcal{W}$ is a Glimm space. See [6] for a characterization of Glimm spaces.

**Lemma 14.** Suppose $\varphi \in B(\mathcal{W})^\#$. The following are equivalent:

1. $\varphi$ is a weak* limit of rank-one tensors.
2. $\varphi$ is completely rank-nonincreasing.

Moreover, if $\mathcal{W}$ is a Glimm space and $\overline{F(\mathcal{W})} = \mathcal{K}(\mathcal{W})$, then every completely rank-nonincreasing map $\phi \in B(\mathcal{W})^\#$ is a weak* limit of bounded rank-one tensors.
Theorem 17. Suppose $\mathcal{F}(\mathcal{W}) = \mathcal{K}(\mathcal{W})$ and $\phi$ is completely rank-nonincreasing. In [11] we proved that $\phi|_{\mathcal{F}(\mathcal{W})} = x \otimes \alpha$, for some $x \in \mathcal{W}$ and $\alpha \in \mathcal{W}^\#$. It is easy to show that $\alpha$ can be chosen to be continuous and $\|x\| = \|\alpha\| \leq \sqrt{\|\phi\|}$. Since $\phi - (x \otimes \alpha)$ annihilates $\mathcal{F}(\mathcal{W})$, then $\phi - (x \otimes \alpha)$ annihilates $\mathcal{F}(\mathcal{W}) = \mathcal{K}(\mathcal{W})$. Therefore there is a bounded net $\{y_\lambda\}$ converging weakly to 0 in $\mathcal{W}$ and a bounded net $\{\beta_\lambda\}$ converging weak* to 0 in $\mathcal{W}^\#$ such that $(\phi - x \otimes \alpha)(T) = \lim_\lambda (y_\lambda \otimes \beta_\lambda)T$ for every $T \in B(\mathcal{W})$. Let $x_\lambda = x + y_\lambda$ and $\alpha_\lambda = \alpha + \beta_\lambda$. Then for every $T \in B(\mathcal{W})$ we have

$$(x_\lambda \otimes \alpha_\lambda)T = (y_\lambda \otimes \beta_\lambda)T + (x \otimes \alpha)T + (y_\lambda \otimes \alpha)T + (x \otimes \alpha_\lambda)T.$$ 

Since $\alpha_\lambda$ converges to 0 in weak* topology, then $(x \otimes \alpha_\lambda)T \to 0$, and since $y_\lambda$ converges weakly to 0, then $(y_\lambda \otimes \alpha)T = \alpha(T(y_\lambda)) = T^\# \alpha(y_\lambda) \to 0$. Therefore, from above, we have

$$\lim_\lambda (x_\lambda \otimes \alpha_\lambda)T = (\phi - x \otimes \alpha)(T) + (x \otimes \alpha)T = \varphi(T).$$

Example 15. Suppose $\mathcal{W}$ is a reflexive Banach space for which $\overline{\mathcal{F}(\mathcal{W})} \neq \mathcal{K}(\mathcal{W})$ (see [2]). We will show that not every completely rank-nonincreasing map $\varphi \in B(\mathcal{W})^\#$ is a bounded limit of rank-one tensors. Let $T_0 \in \mathcal{K}(\mathcal{W}) \setminus \overline{\mathcal{F}(\mathcal{W})}$ and take $\varphi \in B(\mathcal{W})^\#$ such that $\varphi(T_0) = 1$ and $\varphi|_{\mathcal{F}(\mathcal{W})} = 0$. We claim that $\varphi$ is not a bounded limit of rank-one tensors. On the contrary, assume that there exist bounded nets $\{x_\lambda\}$ and $\{\alpha_\lambda\}$ such that $\varphi(T) = \lim_\lambda (x_\lambda \otimes \alpha_\lambda)T$. Since $\mathcal{W}$ is reflexive and $\{x_\lambda\}$ is bounded, by taking a subnet of $\{x_\lambda\}$ if necessary, we can assume that $x_\lambda$ converges to an $x$ weakly and $\alpha_\lambda$ converges to an $\alpha$ in weak* topology. Thus for every operator $T \in \mathcal{K}(\mathcal{W})$, $\|T(x_\lambda) - T(x)\| \to 0$, hence $\alpha_\lambda(T(x_\lambda)) \to \alpha(T(x))$. In other words $\varphi|_{\mathcal{F}(\mathcal{W})} = x \otimes \alpha$. But since $\varphi|_{\mathcal{F}(\mathcal{W})} = 0$, we get that $x \otimes \alpha|_{\mathcal{F}(\mathcal{W})} = 0$, and therefore, $x \otimes \alpha = 0$. This contradicts the fact that $\varphi(T_0) = 1$.

Corollary 16. If $(X, Y, E)$ is a reflexivity triple in which $X = B(\mathcal{W})$, $Y = B(\mathcal{W})^\#$, and

$$E = \{x \otimes \alpha: x \in \mathcal{W}, \alpha \in \mathcal{W}^\# \text{weak}^* \subset Y,$$

and if $\mathcal{G}$ is a norm-closed linear subspace of $B(\mathcal{W})$ such that $\overline{\mathcal{F}(\mathcal{W})} \subset \mathcal{G}$, then $\mathcal{G} \perp + E = E$.

By combining the previous corollary, Theorems 3 and 6, we obtain the following theorem. Note that the following theorem applies when $\mathcal{G} = \overline{\mathcal{F}(\mathcal{W})}$ or $\mathcal{G} = \mathcal{K}(\mathcal{W})$. When $\mathcal{W}$ is a Hilbert space and $\mathcal{G} = \mathcal{K}(\mathcal{W})$, parts (1) and (2) were obtained by D. Hadwin [7, Proposition 7.6], [5, Corollary 6 and Proposition 16], who used much more complicated techniques.

Theorem 17. Suppose $\mathcal{G}$ is a norm-closed linear subspace of $B(\mathcal{W})$ such that $\overline{\mathcal{F}(\mathcal{W})} \subset \mathcal{G}$. Suppose also that $\mathcal{S} \subset B(\mathcal{W})$ is a norm-closed linear space such that $\mathcal{S} + \mathcal{G}$ is norm closed. Then

1. $\mathcal{S} + \mathcal{G}$ is approximately hyperreflexive.
2. $\text{ApprRef}(\mathcal{S} \cap \mathcal{G}) = \text{ApprRef}(\mathcal{S}) \cap \mathcal{G}.$
3. $\text{ApprRef}(\mathcal{S}) = \mathcal{S} + \text{ApprRef}(\mathcal{S} \cap \mathcal{G}).$
4. $\mathcal{S}$ is approximately reflexive if and only if $\mathcal{S} \cap \mathcal{G}$ is.
5. $\mathcal{S}$ is approximately hyperreflexive if and only if $\mathcal{S} \cap \mathcal{G}$ is. In particular, if $\mathcal{S} \cap \mathcal{G} = 0$, then $\mathcal{S}$ is hyperreflexive.
The main theorem of D. Hadwin in [5, Theorem 13] says that every unital C*-algebra \( A \) of \( B(H) \) is approximately hyperreflexive. We prove D. Hadwin's theorem with an elementary method. Note also that, unlike Hadwin's theorem, \( A \) does not have to be unital in our theorem.

**Theorem 18.** If \( A \) is a C*-algebra of \( B(H) \), then \( A \) is approximately hyperreflexive.

**Proof.** Since \( A + \mathcal{K}(H) \) is closed in \( B(H) \), by Theorem 17, it is enough to show that \( A \cap \mathcal{K}(H) \) is approximately hyperreflexive. Since \( A \cap \mathcal{K}(H) \) is a C*-subalgebra of \( \mathcal{K}(H) \), then \( A \cap \mathcal{K}(H) = \bigoplus_i \mathcal{K}(H_i) \) where \( \mathcal{K}(H_i) \) is the space of compact operators on some Hilbert space \( H_i \). It is well known that \( \bigoplus_i \mathcal{B}(H_i) \) is a hyperreflexive subalgebra of \( B(H) \). Hence

\[
\left( \bigoplus_i \mathcal{B}(H_i) \right) \cap \mathcal{K}(H) = \bigoplus_i \mathcal{K}(H_i)
\]

is approximately hyperreflexive by Theorem 17. \( \square \)

**Lemma 19.**

1. If \( S \) is a reflexive subspace of \( B(W) \) and \( A, B \in B(W) \), then the space \( T = \{ T \in B(W): ATB \in S \} \) is also reflexive.

2. If \( A_\lambda \) and \( B_\lambda \) are bounded nets in \( B(W) \), then the space

\[
\mathcal{M} = \{ T \in B(W): \| A_\lambda T B_\lambda \| \to 0 \}
\]

is approximately reflexive.

**Proof.** (1) Suppose \( W \not\in T \). Then \( A W B \not\in S \) and therefore there are operators \( C \) and \( D \) such that \( C S D = \{0\} \) and \( C A W B D \not= 0 \). Thus \( (C A) T (B D) = \{0\} \) and \( (C A) W (B D) \not= 0 \) which implies that \( W \not\in \text{Ref}(T) \). Therefore \( \text{Ref}(T) = \{T\} \).

(2) Let \( S \in \text{Ref}(\mathcal{M}) \). Since \( \| A_\lambda T B_\lambda \| \to 0 \) for every \( T \in \mathcal{M} \), and \( S \in \text{Ref}(\mathcal{M}) \), by definition we have that \( \| A_\lambda S B_\lambda \| \to 0 \). Thus \( S \in \mathcal{M} \). \( \square \)

Recall that an abstract C*-algebra \( B \) is called *elementary* if \( B \) is *-isomorphic to \( \mathcal{K}(H) \) for some Hilbert space \( H \). To prove one of our main results, Theorem 21, we need to show that if \( \mathcal{A} \) is a C*-algebra, and \( \pi : \mathcal{A} \to \mathcal{K}(W) \) is a bounded injective homomorphism, then \( \mathcal{A} \) is *-isomorphic to a C*-direct sum of elementary C*-algebras.

The following result is a key tool in proving our main theorem in this section.

**Proposition 20.** Let \( \mathcal{A} \) be a C*-algebra such that 0 is the only limit point of the spectrum of every Hermitian element in \( \mathcal{A} \). Then \( \mathcal{A} \) is *-isomorphic to a direct sum of elementary C*-algebras.

**Proof.** We can assume that \( \mathcal{A} \subset B(H) \) for some Hilbert space \( H \).

First note that if \( a = a^* \in \mathcal{A} \), then either the \( \sigma(a) \) is finite or \( \sigma(a) = \{\lambda_1, \lambda_2, \ldots\} \) with \( \lambda_n \to 0 \), and, by the spectral theorem, there is an orthogonal sequence \( \{e_k\} \) of projections in \( \mathcal{A} \) such that \( a = \sum_{k=1}^{\infty} \lambda_k e_k \). By considering partial sums, we see that every Hermitian element of \( \mathcal{A} \) can be approximated arbitrarily closely by Hermitian operators with finite spectrum.
It follows from the hypothesis that if \( a = a^* \in A \), and \( \sigma(a) \) is not contained in \( \{0, 1\} \), then there is a continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(0) = 0 = f(1) \) and \( f(a) \) is a projection. We call a projection \( e \in A \) minimal if \( e \not= 0 \) and if the only subprojections of \( e \) in \( A \) are 0 and \( e \). It is now easily shown that \( eAe = \mathbb{C}e \) whenever \( e \) is a minimal projection in \( A \). If \( \{P_n\} \) is a strictly decreasing sequence of projections in \( A \), then 1 is a limit point of the spectrum of

\[
P_1 + \sum_{n=1}^{\infty} \frac{1}{2^n} (P_n - P_{n+1}).
\]

Thus every projection in \( A \) is a finite sum of orthogonal minimal projections in \( A \). Let \( \mathcal{E} \) be the collection of all minimal projections in \( A \). Define a relation \( \sim \) on \( \mathcal{E} \) by

\[
e \sim f \iff eAf \not= \{0\}.
\]

It can be easily shown that \( \sim \) is an equivalence relation. For each \( e \in \mathcal{E} \) let \( \widehat{e} = \{ f \in \mathcal{E} : e \sim f \} \), and let \( \widehat{\mathcal{E}} = \{ \widehat{e}_\lambda : \lambda \in \Lambda \} \) be the set of all of these equivalence classes.

Suppose that \( e, f \in A \) are minimal projections such that \( e \sim f \). Choose \( V_{e,f} \in eAf \) such that \( \|V_{e,f}\|^2 = \|V_{e,f}V_{e,f}^*\| = 1 \). Since \( 0 \leq V_{e,f}V_{e,f}^* \in eAe = \mathbb{C}e \), we know \( V_{e,f}V_{e,f}^* = e \). Similarly, \( V_{e,f}^*V_{e,f} = f \).

If \( x \) is a unit vector in \( e_\lambda(H) \), then \( H_x = [Ax]^\perp \) is a reducing subspace for \( A \). If \( y \) is a unit vector in \( e_\gamma(H) \) and \( y \perp x \), then \( H_y \perp H_x \), since

\[
(Ax, By) = (e_\lambda B^* A e_\lambda x, y) \in \mathbb{C}(e_\lambda x, y) = \{0\}.
\]

It follows that \( \text{rank}(e_\lambda|_{H_x}) \) is a rank-one projection. It also follows, for every \( f \in \widehat{e}_e \), that \( f|H_x = V_{e,f}^* V_{e,f}|H_x \) must also be a rank-one projection. If \( \gamma \in \Lambda \) and \( \gamma \not= \lambda \), then \( e_\gamma(H_x) = [e_\gamma A e_\lambda x]^\perp = \{0\} \). Thus, for any \( f \in \widehat{e}_e \), we have \( f|H_x = 0 \). Hence every minimal projection in \( A \), when restricted to \( H_x \) has rank at most 1. Since \( A \) is the (nonunital) \( C^* \)-algebra generated by its minimal projections,

\[A|_{H_x} = C_0^*(\{f|_{H_x} : f \in \widehat{e}_e\}) \subset K(H_x).
\]

For each \( \lambda \in \Lambda \), choose a unit vector \( x_\lambda \in e_\lambda(H) \) and let \( H_\lambda = H_{x_\lambda} \). Let \( M = \bigoplus_{\lambda \in \Lambda} H_\lambda \), and define \( \pi : A \to B(M) \) by

\[
\pi(T) = T|_M.
\]

Since \( \pi(f) \) is a rank-one projection for each minimal projection in \( A \), we have that

\[
\pi(A) = C_0^*(\{\pi(f) : f \text{ a minimal projection}\}) \subset K(M).
\]

Moreover, if \( T \in A \), there is an orthogonal sequence \( \{f_n\} \) of minimal projections such that \( T^*T = \sum_n \alpha_n f_n \), and \( \pi(T)^* \pi(T) = \sum_n \alpha_n \pi(f_n) \), so \( \|T\|^2 = \|\pi(T)\|^2 = \sup_n \alpha_n \). Thus \( A \) is isomorphic to \( \pi(A) \), which is a \( C^* \)-algebra of compact operators.

D. Hadwin proved in [3] that every \( C^* \)-algebra of operators on a Hilbert space \( \mathcal{H} \) is approximately reflexive, so is every algebra similar to a \( C^* \)-algebra. If Kadison’s similarity problem (Is


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every bounded homomorphism from a C*-algebra into $\mathcal{B}(\mathcal{H})$ similar to a *-homomorphism?) has an affirmative answer, then every bounded unital homomorphism from a C*-algebra into $\mathcal{B}(\mathcal{H})$ would have an approximately reflexive range. We prove more. Note that D. Hadwin and M. Orhon [9] proved the following theorem (in a much more complicated way) in the case where the C*-algebra $\mathcal{A}$ is commutative.

**Theorem 21.** Suppose that $\mathcal{A}$ is a unital C*-algebra, $\mathcal{W}$ is a Banach space, and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{W})$ is a bounded unital homomorphism. Then $\pi(\mathcal{A})$ is approximately reflexive in $\mathcal{B}(\mathcal{W})$.

**Proof.** By replacing $\mathcal{A}$ with $\mathcal{A}/\ker(\pi)$, we can assume that $\pi$ is a one-to-one map. Using an equivalent norm on $\mathcal{W}$ we can assume that $\pi$ is an isometry (see [10, Lemma 2] and the remarks before the lemma for the details). Let $\mathcal{F} = \overline{\mathcal{F}(\mathcal{W})}$ be the closure of finite rank operators in $\mathcal{B}(\mathcal{W})$ and let $\nu : \mathcal{B}(\mathcal{W}) \to \mathcal{B}(\mathcal{W})/\mathcal{F}$ be the quotient map. Then $\nu \circ \pi : \mathcal{A} \to \mathcal{B}(\mathcal{W})/\mathcal{F}$ is a bounded unital homomorphism; hence $\nu \circ \pi(\mathcal{A}) = \pi(\mathcal{A}) + \mathcal{F}$ is closed. In view of Theorem 3, to show that $\pi(\mathcal{A})$ is approximately reflexive, it suffices to show that $\pi(\mathcal{A}) \cap \mathcal{F}$ is approximately reflexive. Note that $\pi^{-1}(\pi(\mathcal{A}) \cap \mathcal{F}) = \mathcal{J}$ is a closed ideal (and so a *-ideal), and $\pi(\mathcal{J}) = \pi(\mathcal{A}) \cap \mathcal{F}$. Thus by replacing $\mathcal{A}$ with $\mathcal{J}$, we can assume that $\pi : \mathcal{A} \to \mathcal{F}$ is an isometry. The fact that 0 is the only limit point of $\sigma(T)$ for every $T \in \mathcal{F}(\mathcal{W})$, implies that 0 is the only limit point of $\sigma(a)$ for every $a \in \mathcal{A}$. Therefore, by Theorem 20, $\mathcal{A} = \bigoplus_i \mathcal{K}(\mathcal{H}_i)$ for some Hilbert spaces $\mathcal{H}_i$. Next, suppose that $\{e_{i,k}\}_{k \geq 1}$ is an orthonormal basis for $\mathcal{H}_i$ and define the projection $P_{i,n} \in \mathcal{K}(\mathcal{H}_i)$ by $P_{i,n} = \sum_{k=1}^n e_{i,k} \otimes e_{i,k}$. It is clear that $\|P_{i,n}AP_{i,n} - A\| \to 0$ for every $A \in \mathcal{K}(\mathcal{H}_i)$ as $n \to \infty$. Let

$$\mathcal{A} = \{(F, n) : F \text{ is finite subset of } \mathcal{I}, \text{ and } n \geq 1\}$$

and make $\mathcal{A}$ into an order set by defining that $(F, n) \leq (G, m)$ if and only if $F \subset G$ and $n \leq m$. For every $\lambda = (F, n) \in \mathcal{A}$ define $P_\lambda$ to be the projection in $\bigoplus_i \mathcal{K}(\mathcal{H}_i)$ whose $i$th coordinate is 0 if $i \notin F$ and is $P_{i,n}$ if $i \in F$. It follows that

$$\|P_\lambda AP_\lambda - A\| \to 0, \quad \text{for every } A \in \mathcal{A} = \bigoplus_i \mathcal{K}(\mathcal{H}_i).$$

Thus

$$\|\pi(P_\lambda)\pi(A)\pi(P_\lambda) - \pi(A)\| \to 0, \quad \forall A \in \mathcal{A}. $$

This shows that if $T \in \text{ApprRef}(\pi(\mathcal{A}))$, then we should also have

$$\|\pi(P_\lambda)T \pi(P_\lambda) - T\| \to 0.$$
Moreover, \( \pi(P_\lambda) \in \mathcal{F} \) is an idempotent and compact and therefore is a finite rank operator. Recall that if \( \varphi: \mathcal{M}_k(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C}) \) is a unital algebra homomorphism, then \( k \) divides \( m \), \( \mathcal{M}_m(\mathbb{C}) = \mathcal{M}_k(\mathcal{M}_{m/k}(\mathbb{C})) \), and \( \varphi(E_{ij}) = \widehat{E}_{ij} \), where \( \widehat{E}_{ij} \) is the matrix having 1 in the \( (i,j) \)-position and zero everywhere else and \( \widehat{E}_{ij} \in \mathcal{M}_k(\mathcal{M}_{m/k}(\mathbb{C})) \) is the matrix having the identity matrix \( I_{m/k} \in \mathcal{M}_{m/k}(\mathbb{C}) \) in the \( (i,j) \)-position and the zero matrix \( 0_{m/k} \in \mathcal{M}_{m/k}(\mathbb{C}) \) everywhere else. From this, it follows that

\[
\text{AlgLat}(\varphi(\mathcal{M}_k(\mathbb{C}))) = \varphi(\mathcal{M}_k(\mathbb{C})).
\]

Thus \( \varphi(\mathcal{M}_k(\mathbb{C})) \) is reflexive, hence approximately reflexive. Since

\[
\pi: P_\lambda AP_\lambda \to \pi(P_\lambda)\mathcal{F}(X)\pi(P_\lambda)
\]

can be viewed as a unital map from a finite direct sum of matrix algebras into a matrix algebra, it follows that \( \pi(P_\lambda AP_\lambda) \) is approximately reflexive. It is also easy to show that \( T \in \text{ApprRef}(\pi(A)) \) implies \( \pi(P_\lambda) T \pi(P_\lambda) \in \text{ApprRef}(\pi(P_\lambda) \pi(A) \pi(P_\lambda)) \). Hence we have

\[
\pi(P_\lambda) T \pi(P_\lambda) \in \text{ApprRef}(\pi(P_\lambda) \pi(A)) = \pi(P_\lambda) \pi(A) \subset \pi(A).
\]

Therefore \( T \in \pi(A) = \pi(\mathcal{A}) \) and so \( \pi(\mathcal{A}) \) is approximately reflexive in \( B(W) \).

4. A new version of algebraic reflexivity

In this section we introduce a version of algebraic reflexivity. Here \( \mathcal{F}(V) \) denotes the space of all finite rank transformations on \( V \).

**Definition 22.** Let \((X, Y, E)\) be a reflexivity triple, where \( X = \mathcal{L}(V) \), \( Y = \mathcal{L}(V)^{\diamond} \), and \( E \subset Y \) is the space of all completely rank-nonincreasing linear maps in \( Y \) and let \( \text{ApprRef}_E(S) = \text{Ref}_E(S) \). A subspace \( S \subset \mathcal{L}(V) \) is called *approximately algebraically reflexive* if \( \text{ApprRef}_E(S) = S \).

Corollary 13 of the preceding section can be translated into a statement that, in view of the definition of \( \text{ApprRef}_E(S) \), justifies the term “approximate algebraic reflexivity.”

**Corollary 23.** In the reflexivity triple \((\mathcal{L}(V), \mathcal{L}(V)^{\diamond}, E)\), if \( E \) is the set of all rank-one tensors \( x \otimes \alpha \) with \( x \in V \) and \( \alpha \in V^{\diamond} \), then \( \overline{E} \) is the set of completely rank-nonincreasing linear functionals on \( \mathcal{L}(V) \).

The following is a corollary of Theorem 8.

**Corollary 24.** For every linear subspace \( S \subset \mathcal{F}(V) \) we have \( \text{ApprRef}_0(S) = \text{Ref}_0(S) \cap \mathcal{F}(V) \).

**Proof.** Let \( \mathcal{F} = \mathcal{F}(V) \) and let \( E_0 \) be the set of rank-one tensors in \( Y = \mathcal{L}(V)^{\diamond} \). This implies that \( \text{Ref}_0(S) = \text{Ref}_{E_0}(S) \). Since every linear map in \( Y \) that kills the finite rank operators is completely rank nonincreasing, we have \( E_0 + \mathcal{F}^{\perp} = E \). The proof is completed by applying Theorem 8.

If \((\mathcal{L}(V), Y, E)\) is an approximately algebraic triple and \( \mathcal{F} = \mathcal{F}(V) \), then it is obvious that \( \mathcal{F}^{\perp} + E = E \) and that every linear subspace of \( X = \mathcal{L}(V) \) is \( \sigma(X, Y) \)-closed. Thus by applying
Theorem 3 we have the following result which is an analogue of results obtained by D. Larson for algebraic reflexivity (with certain countability of dimension restrictions).

**Theorem 25.** Suppose $S$ is a linear subspace of $L(V)$. Then

1. $S + \mathcal{F}(V)$ is approximately algebraically reflexive.
2. $\text{ApprRef}_0(S \cap \mathcal{F}) = \text{ApprRef}_0(S) \cap \mathcal{F}$.
3. $\text{ApprRef}_0(S) = S + \text{ApprRef}_0(S \cap \mathcal{F})$.
4. $S$ is approximately algebraically reflexive if and only if $S \cap \mathcal{F}(V)$ is.

It was shown in [15] that if $S \subset B(V)$ is a finite-dimensional subspace, then $\text{Ref}_0(S)$ is finite-dimensional as well. Since $S \subset \text{ApprRef}_0(S) \subset \text{Ref}_0(S)$, for every subspace $S \subset B(V)$, we can say the same thing for approximate algebraic reflexivity.

It is well known that if a linear subspace $S$ of $L(V)$ has a separating vector, then every linear functional on $S$ can be represented as a rank-one tensor. The following lemma is a generalization of this fact.

**Lemma 26.** Suppose $\{ S_\lambda : \lambda \in \Lambda \}$ is an increasingly directed family of linear subspaces of $L(V)$ such that each $S_\lambda$ has a separating vector, and let $S = \bigcup_{\lambda \in \Lambda} S_\lambda$. Then every linear map $\phi : S \to \mathbb{F}$ can be represented as limit of rank-one tensors.

**Proof.** Let $\Omega = \{ \mu : \mu$ is a finite subset of $S \}$ and turn $\Omega$ into a directed set by inclusion. For every $\mu \in \Omega$, since $\mu \subset S_\lambda$ for some $\lambda$ and since $S_\lambda$ has a separating vector, there are $x_\mu$ and $y_\mu$ in $V$ such that $\phi(S) = (x_\mu \otimes y_\mu)(S)$ for every $S \in \mu$. Now it is clear that $\phi = \lim_{\mu}(x_\mu \otimes y_\mu)$. \qed

Lemma 26 yields the following corollary.

**Corollary 27.** If $S$ is as in Lemma 26 and $S$ is approximately algebraically reflexive, then every linear subspace of $S$ is approximately algebraically reflexive.

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**References**